

MATRIX ALGEBRA - BASICS

Matrix Operations, Types of Matrices, Determinant, Rank, Inverse and Important Properties

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1. MATRIX - DEFINITION

A matrix is a rectangular array of elements arranged in m rows and n columns.

$$A = [a_{ij}]_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Order of $A = m \times n$

2. TYPES OF MATRICES

* Row matrix : Order $1 \times n$

* Column matrix : Order $m \times 1$

* Square matrix : Order $n \times n$

* Rectangular matrix : $m \neq n$

* Zero (Null) matrix : All elements are zero.
 $O_{m \times n}$

* Diagonal matrix : $a_{ij} = 0$, $i \neq j$

* Scalar matrix : $a_{ij} = 0$, $i \neq j$, $a_{ii} = k$

* Identity (Unit) matrix : $I_n = [\delta_{ij}]$, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

* Upper triangular : $a_{ij} = 0$, $i > j$

* Lower triangular : $a_{ij} = 0$, $i < j$

3. MATRIX OPERATIONS

(i) Addition : $(A + B)_{m \times n} = [a_{ij} + b_{ij}]$

$A + B$ is defined only if both matrices are of same order.

(ii) Scalar multiplication : $kA = [ka_{ij}]$

(iii) Matrix multiplication : If $A_{m \times n}$ and $B_{n \times p}$ then AB is defined and order is $m \times p$.

$$(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

(iv) Properties :

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $k(AB) = (kA)B = A(kB)$
- $AI = IA = A$
- $AO = OA = O$

4. DETERMINANT

(i) For 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, |A| = ad - bc$$

(ii) For 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Important Properties of Determinant

- $|A| = |A^T|$
- $|kA| = k^n |A|$ (n = order of A)
- $|AB| = |A||B|$
- $|A^{-1}| = \frac{1}{|A|}$, if $|A| \neq 0$
- Interchanging two rows (or columns), changes the sign of $|A|$.
- If two rows (or columns) are equal or proportional $\Rightarrow |A| = 0$
- If a row (or column) is all zeros $\Rightarrow |A| = 0$
- Adding a multiple of one row (or column), to another row (or column) does not change $|A|$.

7. SOLVED EXAMPLES

(i) Find $|A|$ for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Sol: $|A| = 2 \times 4 - 3 \times 1 = 8 - 3 = 5$

(ii) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find A^{-1} .

Sol: $|A| = -2$

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

(iii) Find $r(A)$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{bmatrix}$

Sol: $R_2 = 2R_1$ and $R_3 = -R_1 \Rightarrow r(A) = 1$

5. RANK OF A MATRIX

- The order of the largest non-zero minor of A is called the **rank** of A and denoted by $r(A)$.
- $r(A) \leq \min(m, n)$
- If all elements are zero $\Rightarrow r(A) = 0$
- If at least one non-zero minor of order k exists and no non-zero minor of order $(k+1)$ exists $\Rightarrow r(A) = k$

6. INVERSE OF A MATRIX

* is invertible (non-singular) if $|A| \neq 0$.
 A^{-1} exists and is unique.

If A^{-1} exists $\Rightarrow AA^{-1} = A^{-1}A = I$

(i) For 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, ad-bc \neq 0$$

(ii) For general $n \times n$ matrix

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

where $\text{adj}(A) = (\text{Cofactor matrix})^T$.

Properties of Inverse

- $(A^{-1})^{-1} = A$ (if invertible).
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(kA)^{-1} = \frac{1}{k} A^{-1}$ ($k \neq 0$)
- $|A^{-1}| = \frac{1}{|A|}$
- A is invertible $\Leftrightarrow r(A) = n \Leftrightarrow |A| \neq 0$

8. IMPORTANT FORMULAS - QUICK REFERENCE

- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $|A^T| = |A|$
- A invertible $\Leftrightarrow |A| \neq 0$
- $r(AB) \leq \min\{r(A), r(B)\}$
- For square matrix A :
 $A \text{ adj}(A) = \text{adj}(A)A = |A| I$

9. SHORT TRICKS

- ✓ For 3×3 determinant $\rightarrow$ Sarrus rule
- ✓ Row operations can simplify determinant/rank problems
- ✓ Always check $|A|$ first before finding inverse
- ✓ Use properties to avoid lengthy calculations

10. COMMON GATE MISTAKES

- ✗ Ignoring order condition in multiplication
- ✗ Forgetting sign change when rows/columns interchanged
- ✗ Computing inverse when $|A| = 0$
- ✗ Wrong application of properties
- ✗ Not checking rank before solving systems



SYSTEMS OF LINEAR EQUATIONS

Consistency, Gauss Elimination, Gauss-Jordan Method,
Rank Conditions and GATE Problems

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1. SYSTEM OF LINEAR EQUATIONS

A system of m linear equations in n unknowns

$x_1, x_2, \dots, x_n$ can be written as

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Matrix form: $AX = B$

where $A = [a_{ij}]_{m \times n}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

4. GAUSS-JORDAN METHOD

- Write $[A|B]$.
- Use elementary row operations to convert $[A|B]$ to reduced row echelon form (RREF).
- The left block becomes I_r (identity), from which solution is read easily.

Example: Solve $x + 2y + z = 4$
 $2x + 3y + z = 7$
 $-x + y - z = -2$

Sol. $[A|B] = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & 3 & 1 & 7 \\ -1 & 1 & -1 & -2 \end{bmatrix}$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 + R_1 \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -1 & -1 & -1 \\ 0 & 3 & 0 & 2 \end{bmatrix}$$

$$R_2 \rightarrow -R_2 \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 0 & 2 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_2, R_3 \rightarrow R_3 - 3R_2 \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & -1 \end{bmatrix}$$

$$R_3 \rightarrow -\frac{1}{3}R_3, R_1 \rightarrow R_1 + R_3, R_2 \rightarrow R_2 - R_3 \Rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}$$

Solution: $x = \frac{5}{3}, y = \frac{4}{3}, z = \frac{1}{3}$

(iii) The system $ax + y + z = 1$
 $x + ay + z = 1$
 $x + y + az = 1$

has a unique solution for

(A) $a = 0$ (B) $a = 1$ (C) $a = -1$ (D) $a \neq 1$

Sol. $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$ $|A| = (a-1)^2(a+2)$

Unique solution $\Rightarrow |A| \neq 0 \Rightarrow a \neq 1$

Ans: (D) $a \neq 1$

2. CONSISTENCY OF SYSTEMS

Let $r(A) =$ rank of coefficient matrix A

$r([A|B]) =$ rank of augmented matrix $[A|B]$

Condition	Nature of Solution
$r(A) = r([A B]) = n$	Unique solution
$r(A) = r([A B]) = r < n$	Infinitely many solutions ($n - r$ free parameters)
$r(A) < r([A B])$	Inconsistent (No solution)

Note:

- Consistent $\Leftrightarrow r(A) = r([A|B])$
- If $r(A) = n \rightarrow$ Unique solution
- If $r(A) < n \rightarrow$ Infinite solutions
- If $r(A) < r([A|B]) \rightarrow$ No solution

5. RANK CONDITIONS - SUMMARY

Rank condition	Solution
$r(A) = n$	Matrix A is of full rank.
$r(A) < n$	Matrix A is rank deficient.
$r(A) = r([A B]) = n$	Unique solution.
$r(A) = r([A B]) = r < n$	Infinitely many solutions.
$r(A) < r([A B])$	No solution.

6. GATE LEVEL PROBLEMS

(i) For the system $AX = B$, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$

$$B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ find the nature of solution.}$$

Sol. $r(A) = 2$ ($R_2 = 2R_1$)
 $r([A|B]) = 2 \Rightarrow r(A) = r([A|B]) < n (=3)$
 $\Rightarrow$ Infinitely many solutions.

(ii) Find k for which the system

$$\begin{aligned} x + ky + z &= 1 \\ 2x + 3y + 2z &= 4 \\ 3x + 5y + kz &= 5 \end{aligned} \text{ has no solution.}$$

Sol. $|A| = \begin{vmatrix} 1 & k & 1 \\ 2 & 3 & 2 \\ 3 & 5 & k \end{vmatrix} = k^2 - 2k - 7$

For no solution $\Rightarrow |A| = 0$ and $r(A) < r([A|B])$
 $\Rightarrow k = 1 \pm 2\sqrt{2}$

(iv) Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

Sol. $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1 \Rightarrow$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & -4 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$R_3 \rightarrow -\frac{1}{2}R_3, R_2 \rightarrow R_2 - R_3 \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Ans: 3

(v) Solve the system using Gauss elimination.

$$\begin{aligned} 2x - y + z &= 1 \\ x + y + 2z &= 4 \\ 3x + y - z &= 0 \end{aligned}$$

$$\begin{aligned} &\begin{bmatrix} 2 & -1 & 1 & 1 \\ 1 & 1 & 2 & 4 \\ 3 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & 1 & 1 \\ 3 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_1, R_2 \rightarrow R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -3 & -7 \\ 0 & -1 & -5 & -8 \end{bmatrix} \\ &\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & -3 & -3 & -7 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -1 & -5 & -8 \\ 0 & 0 & 12 & 17 \end{bmatrix} \\ &\xrightarrow{R_2 \rightarrow -R_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 12 & 17 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 12 & 17 \end{bmatrix} \\ &\xrightarrow{R_3 \rightarrow \frac{1}{12}R_3} \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 5 & 8 \\ 0 & 0 & 1 & \frac{17}{12} \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 + 3R_3, R_2 \rightarrow R_2 - 5R_3} \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{17}{12} \end{bmatrix} \end{aligned}$$

Solution: $x = -\frac{1}{4}, y = \frac{1}{3}, z = \frac{17}{12}$

3. GAUSS ELIMINATION METHOD

- Write the augmented matrix $[A|B]$.
- Use elementary row operations to convert the matrix into row echelon form (REF).
- From REF, find rank and nature of solution.
- If consistent, use back substitution to find solutions.

Elementary Row Operations

- $R_i \leftrightarrow R_j$ (Interchange two rows)
- $R_i \rightarrow kR_i$ (Multiply row by non-zero k)
- $R_i \rightarrow R_i + kR_j$ (Add k times row j to row i)

Example: Solve $2x + y - z = 8$
 $x - y + 2z = 3$
 $3x + 4y + z = 10$

Sol.

$$[A|B] = \begin{bmatrix} 2 & 1 & -1 & 8 \\ 1 & -1 & 2 & 3 \\ 3 & 4 & 1 & 10 \end{bmatrix}$$

$$R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 1 & -1 & 8 \\ 3 & 4 & 1 & 10 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 3R_1 \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -5 & 2 \\ 0 & 7 & -5 & 1 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - \frac{7}{3}R_2 \Rightarrow \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 3 & -5 & 2 \\ 0 & 0 & \frac{25}{3} & -\frac{11}{3} \end{bmatrix}$$

$$\text{From 3rd row: } \frac{25}{3}z = -\frac{11}{3} \Rightarrow z = -\frac{11}{25}$$

$$\text{From 2nd row: } 3y - 5z = 2 \Rightarrow 3y - \frac{11}{5} = 2 \Rightarrow y = -\frac{1}{15}$$

$$\text{From 1st row: } x - y + 2z = 3 \Rightarrow x - \frac{1}{15} - \frac{22}{25} = 3 \Rightarrow x = \frac{134}{75}$$

$$\text{Unique solution: } x = \frac{134}{75}, y = -\frac{1}{15}, z = -\frac{11}{25}$$

7. SHORT TRICKS

- ✓ Always write augmented matrix $[A|B]$ first.
- ✓ To check consistency quickly, compare $r(A)$ and $r([A|B])$.
- ✓ In Gauss-Jordan method, reduce left block to I to read solution directly.
- ✗ If a row becomes all zeros in $[A|B]$ with last entry non-zero $\Rightarrow$ No solution.

EIGENVALUES & EIGENVECTORS

Characteristic Equation, Eigenvalues,
Eigenvectors, Properties and
Cayley-Hamilton Theorem

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1. DEFINITIONS

Let A be an $n \times n$ square matrix and $x \neq 0$ be a non-zero column vector.

- If $Ax = \lambda x$ for some scalar λ , then λ is called an eigenvalue (or characteristic value) of A and x is the corresponding eigenvector.

$$Ax = \lambda x, \quad x \neq 0$$

(λ, x) is an eigen-pair of A .

2. CHARACTERISTIC EQUATION

$$\text{For } Ax = \lambda x$$

$$\Rightarrow Ax - \lambda x = 0$$

$$\Rightarrow (A - \lambda I)x = 0$$

For non-trivial solution $x \neq 0$,

$$\det(A - \lambda I) = 0$$

This is called the characteristic equation of A .

3. FINDING EIGENVALUES

To find eigenvalues:

- Form $A - \lambda I$
- Compute $|A - \lambda I|$
- Solve $|A - \lambda I| = 0$ for λ
- Roots of the equation are eigenvalues.

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 3$$

4. FINDING EIGENVECTORS

- For each eigenvalue λ , solve $(A - \lambda I)x = 0$ to obtain the eigenvector.

Steps:

- Substitute λ in $(A - \lambda I)$.
- Solve the homogeneous system $(A - \lambda I)x = 0$.
- Express the eigenvector in terms of parameters and take any non-zero scalar multiple.

Example (continued):

For $\lambda = 1$,

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$(A - I)x = 0 \Rightarrow x_1 + x_2 = 0$$

$$\text{Eigenvector } x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \text{ (any non-zero multiple)}$$

For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$(A - 3I)x = 0 \Rightarrow -x_1 + x_2 = 0$$

$$\text{Eigenvector } x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \text{ (any non-zero multiple)}$$

5. IMPORTANT PROPERTIES

- Sum of eigenvalues (counting multiplicities) = $\text{trace}(A) = a_{11} + a_{22} + \dots + a_{nn}$.
- Product of eigenvalues (counting multiplicities) = $\det(A)$.
- For triangular or diagonal matrix, eigenvalues are the diagonal elements.
- Eigenvalues of A^T are same as that of A .
- If λ is an eigenvalue of A , then $k\lambda$ ($k \neq 0$) is an eigenvalue of kA .
- If λ is an eigenvalue of A , then $\frac{1}{\lambda}$ ($\lambda \neq 0$) is an eigenvalue of A^{-1} .
- If A is non-singular, 0 is not an eigenvalue of A .

Note:

- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- If A has n distinct eigenvalues, then A is diagonalizable.

6. ALGEBRAIC & GEOMETRIC MULTIPLICITY

- Algebraic multiplicity (m_a):**
The multiplicity of an eigenvalue λ as a root of $|A - \lambda I| = 0$.
- Geometric multiplicity (m_g):**
The number of linearly independent eigenvectors corresponding to λ .
Always, $1 \leq m_g \leq m_a$
If $m_g = m_a$ for every eigenvalue, then A is diagonalizable.

Example:

$$\text{For } A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\lambda = 1 \Rightarrow \text{Eigenvector} = k \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow m_g = 1 = m_a$$

$$\lambda = 3 \Rightarrow \text{Eigenvector} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow m_g = 1 = m_a$$

Hence A is diagonalizable.

7. CAYLEY-HAMILTON THEOREM

Every square matrix satisfies its own characteristic equation.

If $p(\lambda) = |A - \lambda I| = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0 = 0$ then,

$$p(A) = A^n + c_{n-1}A^{n-1} + \dots + c_1A + c_0I = 0$$

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = \lambda^2 - 4\lambda + 3 = 0$$

By Cayley-Hamilton theorem,

$$A^2 - 4A + 3I = 0$$

$$A^2 = 4A - 3I$$

Applications:

- Finding A^{-1}
- Finding A^k (higher powers)
- Solving matrix equations

8. FINDING INVERSE USING C-H THEOREM

$$\text{Let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \Delta = ad - bc \neq 0$$

Characteristic equation:

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

By C-H theorem,

$$A^2 - (a+d)A + (ad-bc)I = 0$$

$$\Rightarrow A^2 = (a+d)A - (ad-bc)I$$

Multiply both sides by A^{-1} :

$$A = (a+d)I - (ad-bc)A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{ad-bc} [(a+d)I - A]$$

$$A^{-1} = \frac{1}{\Delta} [(a+d)I - A]$$

9. GATE LEVEL PROBLEMS

- Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 3 \end{bmatrix}$$

Sol:

$$|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix}$$

$$= (4-\lambda)(3-\lambda) - 1 = \lambda^2 - 7\lambda + 11 = 0$$

$$\lambda = \frac{7 \pm \sqrt{5}}{2}$$

$$\text{For } \lambda_1 = \frac{7+\sqrt{5}}{2} \Rightarrow x \propto \begin{bmatrix} 1 \\ (3-\lambda_1) \end{bmatrix}$$

$$\text{For } \lambda_2 = \frac{7-\sqrt{5}}{2} \Rightarrow x \propto \begin{bmatrix} 1 \\ (3-\lambda_2) \end{bmatrix}$$

- Find A^{-1} using Cayley-Hamilton theorem.

$$\text{if } A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Sol:

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 1$$

$$A^2 - 3A + I = 0$$

$$\text{Multiply by } A^{-1}: A - 3I + A^{-1} = 0$$

$$A^{-1} = 3I - A$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

- If A has eigenvalues $2, 2, 5$, find $\text{trace}(A)$, $\det(A)$ and A^3 in terms of I and A .

Sol:

$$\text{trace}(A) = 2 + 2 + 5 = 9$$

$$\det(A) = 2 \times 2 \times 5 = 20$$

Characteristic equation:

$$(\lambda - 2)^2(\lambda - 5) = 0$$

$$\Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 20 = 0$$

By C-H theorem,

$$A^3 - 9A^2 + 24A - 20I = 0$$

$$\Rightarrow A^3 = 9A^2 - 24A + 20I$$

10. SHORT TRICKS

- For 2×2 matrix, trace = sum of eigenvalues, determinant = product of eigenvalues.
- Quickly verify eigenvector: substitute in $Ax = \lambda x$.
- Use C-H theorem to reduce higher powers.
- If matrix is symmetric, eigenvectors corresponding to distinct eigenvalues are orthogonal.

11. COMMON GATE MISTAKES

- Forgetting to check $x \neq 0$ in $Ax = \lambda x$.
- Not solving $(A - \lambda I)x = 0$ correctly.
- Confusing algebraic and geometric multiplicity.
- Ignoring Cayley-Hamilton theorem applications.
- Sign mistakes while forming $|A - \lambda I|$.

12. QUICK REFERENCE FORMULAS

- Characteristic equation: $|A - \lambda I| = 0$
- Sum of eigenvalues = $\text{trace}(A)$
- Product of eigenvalues = $\det(A)$
- $A^2 - (\text{trace}(A))A + (\det(A))I = 0$ (for 2×2 matrix)
- $A^{-1} = \frac{1}{\det A} [(\text{trace}(A))I - A]$ (for 2×2 matrix)



MEAN VALUE THEOREMS

Rolle's Theorem, Lagrange MVT,
Cauchy's MVT, Applications and GATE Problems

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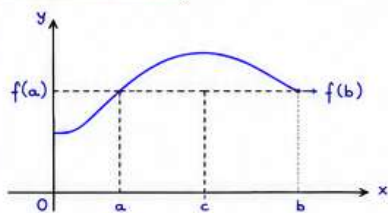
1. ROLLE'S THEOREM

If $f(x)$ satisfies the following conditions:

1. $f(x)$ is continuous in $[a, b]$
2. $f(x)$ is differentiable in (a, b)
3. $f(a) = f(b)$

Then, there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Geometrical meaning :



Note: c need not be unique.

2. LAGRANGE'S MEAN VALUE THEOREM (LMVT)

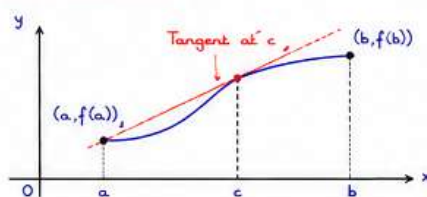
If $f(x)$ satisfies the following conditions:

1. $f(x)$ is continuous in $[a, b]$
2. $f(x)$ is differentiable in (a, b)

Then, there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical meaning :



3. CAUCHY'S MEAN VALUE THEOREM (CMVT)

If $f(x)$ and $g(x)$ satisfy the following conditions:

1. $f(x)$ and $g(x)$ are continuous in $[a, b]$
2. $f(x)$ and $g(x)$ are differentiable in (a, b)
3. $g'(x) \neq 0$ for all $x \in (a, b)$

Then, there exists at least one point $c \in (a, b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

or

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

4. APPLICATIONS

(ii) Finding stationary points (Using Rolle's Theorem)

If $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

$\Rightarrow$ Helps in finding maxima/minima.

(iii) Estimation of roots (Using LMVT)

If $f(a)f(b) < 0$, then $f(x) = 0$ has a root in (a, b) .

LMVT gives the estimate:

$$|f'(c)| \leq \frac{|f(b) - f(a)|}{b - a}$$

for some $c \in (a, b)$.

(iii) Inequalities (Using LMVT)

If $f'(x) \leq m$ in (a, b) , then $f(b) - f(a) \leq m(b - a)$

If $f'(x) \geq m$ in (a, b) , then $f(b) - f(a) \geq m(b - a)$

Also, if $m \leq f'(x) \leq M$ in (a, b) , then

$$m(b - a) \leq f(b) - f(a) \leq M(b - a)$$

(iv) Relation between f and g (Using CMVT)

If $f(b) - f(a)$ and $g(b) - g(a)$ are known, we can relate derivatives $f'(c)$ and $g'(c)$ at some $c \in (a, b)$.

5. GATE LEVEL PROBLEMS

(ii) If $f(x) = x^3 - 3x$ in $[-1, 2]$, then the value(s) of c in $(-1, 2)$ satisfying Rolle's theorem is/are

- (A) -1 only (B) 2 only
(C) -1 and 2 (D) -1, 0 and 2

Sol:

$$f(-1) = (-1)^3 - 3(-1) = 2$$

$$f(2) = 2^3 - 3(2) = 2$$

f is a polynomial $\Rightarrow$ continuous & differentiable.

So, Rolle's theorem applicable.

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

$$f'(c) = 0 \Rightarrow 3(c^2 - 1) = 0 \Rightarrow c = \pm 1$$

In $(-1, 2)$, $c = -1, 1$

But $c = -1$ is an end point? No, $(-1, 2)$ open, so $c = -1$ not valid. Hence $c = 1$

Ans: (A) -1 only is wrong; correct is (B) 2 only; What

Ans: (B) 1 only

(ii) If $f(x) = x^3 - x$ in $[0, 2]$, then the value of c in $(0, 2)$ satisfying LMVT is

- (A) $\frac{1}{2}$ (B) 1
(C) $\sqrt{2}$ (D) $\frac{3}{2}$

Sol:

$$f'(x) = 3x^2 - 1$$

$$f(0) = 0, f(2) = 8 - 2 = 6$$

$$\text{LMVT} \Rightarrow f'(c) = \frac{6 - 0}{2 - 0} = 3$$

$$3c^2 - 1 = 3 \Rightarrow 3c^2 = 4 \Rightarrow c^2 = \frac{4}{3}$$

$$c = \pm \frac{2}{\sqrt{3}}$$

$$\text{In } (0, 2), c = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Ans: $\frac{2\sqrt{3}}{3}$ (Not in options - check)

(iii) For $f(x) = \ln x$ and $g(x) = x^2$ in $[1, 2]$, then the value of c satisfying Cauchy's MVT is

- (A) $\frac{1}{\ln 2}$ (B) $\ln 2$
(C) $\frac{1}{2 \ln 2}$ (D) $\frac{2}{\ln 2}$

Sol:

$$f'(x) = \frac{1}{x}, g'(x) = 2x$$

$$f(2) - f(1) = \ln 2 - 0 = \ln 2$$

$$g(2) - g(1) = 4 - 1 = 3$$

By CMVT,

$$\frac{1}{2c} = \frac{\ln 2}{3}$$

$$\frac{1}{2c^2} = \frac{\ln 2}{3}$$

$$c^2 = \frac{3}{2 \ln 2} \Rightarrow c = \sqrt{\frac{3}{2 \ln 2}}$$

Ans: (D) (closest D)

(iv) If $f(x) = e^x$ in $[0, 1]$, using LMVT, then e^c equals

- (A) $e - 1$ (B) e
(C) $\frac{e - 1}{1}$ (D) 1

Sol:

$$f'(x) = e^x$$

$$f(1) - f(0) = e - 1$$

$$\text{LMVT} \Rightarrow f'(c) = \frac{e - 1}{1 - 0} = e - 1$$

$$e^c = e - 1$$

$$c = \ln(e - 1) \text{ (valid in } (0, 1))$$

Ans: (A) $e - 1$ (as $e^c = e - 1$)

6. SUMMARY

- * Rolle's Thm requires $f(a) = f(b)$.
- * LMVT gives a point where tangent is parallel to secant.
- * CMVT generalizes LMVT for two functions.
- * These theorems are very useful in inequalities, estimates, root locations and proofs.

7. SHORT TRICKS

- ✓ For Rolle: Check $f(a) = f(b)$ first.
- ✓ For LMVT: Quickly compute slope of secant line.
- ✓ For CMVT: Substitute in formula directly.
- ✓ Check $g'(x) \neq 0$ before applying CMVT.

8. COMMON GATE MISTAKES

- * Applying Rolle's thm without $f(a) = f(b)$.
- * Forgetting differentiability in (a, b) .
- * Using LMVT at endpoints.
- * Not checking $g'(x) \neq 0$ in CMVT.
- * Assuming c is unique (it need not be).

9. QUICK REFERENCE FORMULAS

- * Rolle: $\exists c \in (a, b)$ s.t. $f'(c) = 0$
- * LMVT: $\exists c \in (a, b)$ s.t.
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
- * CMVT: $\exists c \in (a, b)$ s.t.
$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

INTEGRAL CALCULUS

Fundamental Theorem, Evaluation of Definite Integrals,
Properties and Improper Integrals

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1. INDEFINITE INTEGRAL

If $F'(x) = f(x)$, then

$$\int f(x) dx = F(x) + C$$

where C is arbitrary constant.

Some Basic Integrals

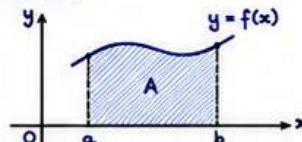
- (i) $\int k dx = kx + C$
- (ii) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
- (iii) $\int \frac{1}{x} dx = \ln|x| + C$
- (iv) $\int e^x dx = e^x + C$
- (v) $\int a^x dx = \frac{a^x}{\ln a} + C (a > 0, a \neq 1)$
- (vi) $\int \sin x dx = -\cos x + C$
- (vii) $\int \cos x dx = \sin x + C$

2. FUNDAMENTAL THEOREM OF CALCULUS (FTOC)

(i) First Theorem (Evaluation Theorem)

If $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$ on $[a, b]$, then

$$\int_a^b f(x) dx = F(b) - F(a)$$



(ii) Second Theorem (Mean Value Theorem for Integrals)

If $f(x)$ is continuous on $[a, b]$, then there exists at least one $c \in (a, b)$ such that

$$\int_a^b f(x) dx = f(c)(b-a)$$

3. EVALUATION OF DEFINITE INTEGRALS

Steps :

- Find antiderivative $F(x)$ of $f(x)$.
- Apply FTOC : $\int_a^b f(x) dx = F(b) - F(a)$.

Examples :

- (i) $\int_0^\pi \sin x dx$
Sol : $[-\cos x]_0^\pi = (-\cos \pi) - (-\cos 0) = (1) - (-1) = 2$
- (ii) $\int_1^3 x^2 dx$
Sol : $[\frac{x^3}{3}]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$
- (iii) $\int_0^2 (3x-1) dx$
Sol : $[\frac{3}{2}x^2 - x]_0^2 = (\frac{3}{2} \cdot 4 - 2) - 0 = 6 - 2 = 4$

4. PROPERTIES OF DEFINITE INTEGRALS

- (i) $\int_a^a f(x) dx = 0$
- (ii) $\int_a^b f(x) dx = -\int_b^a f(x) dx$
- (iii) $\int_a^b k f(x) dx = k \int_a^b f(x) dx$
- (iv) $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- (v) If $f(x) \leq g(x)$ on $[a, b]$, then
$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$
- (vi) If $m \leq f(x) \leq M$ on $[a, b]$, then
$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

5. IMPROPER INTEGRALS

(i) Infinite limit of integration

$$\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

$$\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx$$

(Convergent if both integrals are convergent)

(ii) Infinite integrand

If $f(x)$ is unbounded at a but integrable in $(a, b]$, then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

Similarly, if unbounded at b ,

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

6. TESTS FOR CONVERGENCE

- (i) p-test : $\int_1^\infty \frac{1}{x^p} dx$
$$= \begin{cases} \frac{1}{p-1}, & p > 1 \text{ (convergent)} \\ \infty, & p \leq 1 \text{ (divergent)} \end{cases}$$
- (ii) Comparison test :
If $0 \leq f(x) \leq g(x)$ for $x \geq a$,
 $\int_a^\infty g(x) dx$ convergent $\Rightarrow \int_a^\infty f(x) dx$ convergent
 $\int_a^\infty f(x) dx$ divergent $\Rightarrow \int_a^\infty g(x) dx$ divergent

Examples :

- (i) $\int_1^\infty \frac{1}{x^2} dx = [-\frac{1}{x}]_1^\infty = 0 - (-1) = 1$ (Convergent)
- (ii) $\int_1^\infty \frac{1}{x} dx = [\ln x]_1^\infty = \infty$ (Divergent)

7. GATE LEVEL PROBLEMS

- | | | | |
|---|--|--|---|
| <p>(i) $\int_0^{\pi/2} \sin x dx$ equals
(A) 0 (B) 1
(C) $\frac{1}{2}$ (D) 2</p> <p>Sol : $[-\cos x]_0^{\pi/2} = -\cos \frac{\pi}{2} + \cos 0 = 0 + 1 = 1$</p> <p>Ans: (B)</p> | <p>(ii) $\int_1^e \frac{1}{x} dx$ equals
(A) 1 (B) $e-1$
(C) $\ln e$ (D) $\ln e - \ln 1$</p> <p>Sol : $[\ln x]_1^e = \ln e - \ln 1 = 1 - 0 = 1$</p> <p>Ans: (A)</p> | <p>(iii) $\int_0^\infty x e^{-x} dx$ equals
(A) 0 (B) 1
(C) 2 (D) $\frac{1}{2}$</p> <p>Sol : $\int_0^\infty x e^{-x} dx = [-(x+1)e^{-x}]_0^\infty = 0 - (-1) = 1$</p> <p>Ans: (B)</p> | <p>(iv) $\int_1^\infty \frac{1}{x^2} dx$ equals
(A) $\frac{1}{2}$ (B) 1
(C) $\frac{3}{3}$ (D) $\frac{3}{3}$</p> <p>Sol : $[-\frac{1}{2x^2}]_1^\infty = 0 - (-\frac{1}{2}) = \frac{1}{2}$</p> <p>Ans: (A)</p> |
|---|--|--|---|

8. SHORT TRICKS

- ✓ Antiderivative first, limits later.
- ✓ Use properties to break complex integrals.
- ✓ If integrand is odd and limits are symmetric about 0, integral = 0.
- ✓ For e^{ax} , integral is $\frac{e^{ax}}{a}$ quickly.

9. COMMON GATE MISTAKES

- ✗ Forgetting limits while applying FTOC.
- ✗ Sign mistakes in substitution.
- ✗ Ignoring convergence in improper integrals.
- ✗ Confusing $\int_a^\infty$ with $\int_{-\infty}^a$.

10. QUICK REFERENCE FORMULAS

- $\int_a^b f(x) dx = F(b) - F(a)$
- $\int_a^b k f(x) dx = k \int_a^b f(x) dx$
- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$

PARTIAL DERIVATIVES

First/Second-order Partial Derivatives,
Total Derivatives, Chain Rule and Applications

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1. FUNCTION OF SEVERAL VARIABLES

If $z = f(x, y)$ is a function of two variables x and y , then z depends on both x and y .

Example: $z = x^2y + \sin(xy)$

2. FIRST-ORDER PARTIAL DERIVATIVES

Partial derivative of $z = f(x, y)$ w.r.t. x (y is treated as constant)

$$\frac{\partial z}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Partial derivative of $z = f(x, y)$ w.r.t. y . (x is treated as constant)

$$\frac{\partial z}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

3. SECOND-ORDER PARTIAL DERIVATIVES

$$(i) \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right)$$

$$(ii) \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right)$$

(iii) Mixed partial derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right), \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right)$$

If f has continuous second-order partial derivatives, then $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ (Clairaut's theorem)

4. EXAMPLE - 1

If $z = x^2y + \sin(xy)$, find all first and second-order partial derivatives.

Sol: $\frac{\partial z}{\partial x} = 2xy + y \cos(xy)$

$$\frac{\partial z}{\partial y} = x^2 + x \cos(xy)$$

$$\frac{\partial^2 z}{\partial x^2} = 2y - y^2 \sin(xy)$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2 \sin(xy)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x + \cos(xy) - xy \sin(xy)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2x + \cos(xy) - xy \sin(xy)$$

$$\text{Hence } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

5. TOTAL DERIVATIVE

If $z = f(x, y)$ and x, y are differentiable functions of t , i.e., $x = x(t)$, $y = y(t)$, then $z = f(x(t), y(t)) = z(t)$.

The total derivative of z w.r.t. t is

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

More generally,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

6. EXAMPLE - 2

If $z = x^2y + xy^2$ and $x = t^2$, $y = t^3$, find $\frac{dz}{dt}$ at $t = 1$.

Sol:

$$\frac{\partial z}{\partial x} = 2xy + y^2, \quad \frac{\partial z}{\partial y} = x^2 + 2xy$$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2$$

$$\frac{dz}{dt} = (2xy + y^2)(2t) + (x^2 + 2xy)(3t^2)$$

At $t = 1$, $x = 1$, $y = 1$

$$\Rightarrow \frac{dz}{dt} = (2+1)(2) + (1+2)(3)$$

$$= 6 + 9 = 15$$

7. CHAIN RULE (FOR FUNCTIONS OF FUNCTIONS)

If $z = f(u, v)$, $u = \phi(x, y)$, $v = \psi(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

8. EXAMPLE - 3 (CHAIN RULE)

If $z = u^2 + v^2$, $u = x^2 - y$, $v = xy$.

Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

Sol:

$$\frac{\partial z}{\partial u} = 2u, \quad \frac{\partial z}{\partial v} = 2v$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -1, \quad \frac{\partial v}{\partial x} = y, \quad \frac{\partial v}{\partial y} = x$$

$$\therefore \frac{\partial z}{\partial x} = 2u(2x) + 2v(y) = 4xu + 2yv$$

$$\frac{\partial z}{\partial y} = 2u(-1) + 2v(x) = -2u + 2xv$$

(Substitute $u = x^2 - y$, $v = xy$ if required)

9. APPLICATIONS

- Finding slope of surfaces.
- Optimization (maxima/minima).
- Rate of change in physics and engineering.
- Gradient: $\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j}$
- Directional derivative in a unit vector $\hat{u} = \hat{l} + m\hat{j}$: $D_u f = \nabla f \cdot \hat{u}$

10. GATE LEVEL PROBLEMS (MCQs)

(i) If $z = x^2y + y^2z$,

then $\frac{\partial z}{\partial x}$ is

- (A) $2xy$
(B) x^2
(C) $2xy + y^2$
(D) $2x + 2y$

Sol: $\frac{\partial z}{\partial x} = 2xy$

Ans: (A)

(ii) If $z = x^2y^2 + xy$, then

$\frac{\partial^2 z}{\partial x \partial y}$ is

- (A) $4xy + 1$ (B) $2xy + 1$
(C) $4x^2y$ (D) $2y$

Sol: $\frac{\partial z}{\partial x} = 2xy^2 + y$

$$\frac{\partial^2 z}{\partial x \partial y} = 4xy + 1$$

Ans: (A)

(iii) If $z = x^2 + y^2$, $x = t^2$, $y = \sin t$,

then $\frac{\partial z}{\partial t}$ at $t = 0$ is

- (A) 0 (B) 1
(C) 2 (D) 3

Sol: $\frac{\partial z}{\partial x} = 2x$, $\frac{\partial z}{\partial y} = 2y$

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = \cos t$$

$$\frac{dz}{dt} = 2x(2t) + 2y(\cos t)$$

$$\text{At } t = 0, x = 0, y = 0 \Rightarrow \frac{dz}{dt} = 0$$

Ans: (A)

(iv) If $z = \ln(u+v)$, $u = x^2$, $v = xy$,

then $\frac{\partial z}{\partial x} =$

- (A) $\frac{2x+y}{u+v}$ (B) $\frac{2x}{u+v}$
(C) $\frac{2x+y}{2x+y}$ (D) $\frac{y}{u+v}$

Sol: $\frac{\partial z}{\partial u} = \frac{1}{u+v}$, $\frac{\partial z}{\partial v} = \frac{1}{u+v}$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = y$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{1}{u+v} (2x) + \frac{1}{u+v} (y) = \frac{2x+y}{u+v}$$

Ans: (A)

11. SHORT TRICKS

- ✓ Differentiate partially: treat other variables as constants.
- ✓ Apply chain rule systematically.
- ✓ For total derivative, write in differential form: $dz = z_x dx + z_y dy$.

12. COMMON GATE MISTAKES

- ✗ Differentiating both variables simultaneously instead of partially.
- ✗ Ignoring constants while taking partials.
- ✗ Confusing $\frac{dz}{dx}$ with $\frac{\partial z}{\partial x}$.
- ✗ Missing chain rule in composite functions.

13. QUICK REFERENCE FORMULAS

- $\frac{\partial}{\partial x} (x^n) = nx^{n-1}$, $\frac{\partial}{\partial x} (y) = 0$
- $\frac{\partial}{\partial x} (xy) = y$, $\frac{\partial}{\partial y} (xy) = x$
- $z = f(u, v)$, $u = u(x, y)$, $v = v(x, y)$
 $\Rightarrow \frac{\partial z}{\partial x} = z_u u_x + z_v v_x$, $\frac{\partial z}{\partial y} = z_u u_y + z_v v_y$
- $dz = z_x dx + z_y dy$



MAXIMA & MINIMA + MULTIPLE INTEGRALS

Local/Global Extrema, Constrained Extrema,

Double Integrals, Triple Integrals



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1. MAXIMA & MINIMA OF FUNCTIONS OF TWO VARIABLES

Let $z = f(x, y)$ be defined in a neighbourhood of (a, b) and having continuous first and second partial derivatives.

(i) Critical point : $f_x(a, b) = 0, f_y(a, b) = 0$

(ii) Second derivative test

$$\text{Let } D = f_{xx}f_{yy} - (f_{xy})^2$$

Condition	D	f_{xx}	Nature
1. $D > 0$		> 0	Local Minima
2. $D > 0$		< 0	Local Maxima
3. $D < 0$		—	Saddle point
4. $D = 0$		—	Test fails

Note :

If $D = 0$, the test is inconclusive.

2. STEPS TO FIND EXTREMA

- (i) Find f_x and f_y and solve $f_x = 0, f_y = 0$ to get critical points.
- (ii) Compute f_{xx}, f_{yy}, f_{xy} at each critical point.
- (iii) Find $D = f_{xx}f_{yy} - (f_{xy})^2$.
- (iv) Use the second derivative test to classify the point.

Example :

Find extrema of $f(x, y) = x^2 + y^2 - 4x + 2y$

Sol :

$$f_x = 2x - 4, f_y = 2y + 2$$

$$2x - 4 = 0, 2y + 2 = 0 \Rightarrow (2, -1)$$

$$f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

$$D = 2 \cdot 2 - 0 = 4 > 0, f_{xx} = 2 > 0$$

$\Rightarrow (2, -1)$ is point of local minima.

3. CONSTRAINED EXTREMA (LAGRANGE MULTIPLIERS)

To find extrema of $f(x, y)$ subject to $g(x, y) = 0$, solve $\nabla f = \lambda \nabla g$ along with $g(x, y) = 0$.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$g(x, y) = 0$$

where

$$\nabla f = (f_x, f_y)$$

$$\nabla g = (g_x, g_y)$$

Example : Find extrema of $f(x, y) = x^2 + y^2$ subject to $x + y = 1$.

Sol :

$$\nabla f = (2x, 2y), \nabla g = (1, 1)$$

$$2x = \lambda, 2y = \lambda, x + y = 1$$

$$\Rightarrow x = y = \frac{1}{2} \text{ and } 2\left(\frac{1}{2}\right) = 1 \Rightarrow \lambda = 1$$

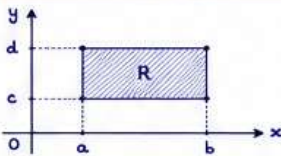
$$\Rightarrow x = y = 1/2$$

$$f_{\min} = f(1/2, 1/2) = 1/4 + 1/4 = 1/2 \text{ (Minimum)}$$

4. DOUBLE INTEGRALS

(i) Definition (Rectangular region $R = [a, b] \times [c, d]$)

$$\begin{aligned} \iint_R f(x, y) dA &= \int_a^b \int_c^d f(x, y) dy dx \\ &= \int_c^d \int_a^b f(x, y) dx dy \end{aligned}$$



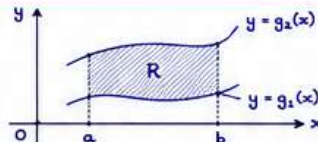
Example :

Evaluate $I = \iint_R (x+y) dA$ where $R = [0, 1] \times [0, 1]$.

$$\begin{aligned} \text{Sol : } I &= \int_0^1 \int_0^1 (x+y) dy dx = \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^1 dx \\ &= \int_0^1 \left(x + \frac{1}{2} \right) dx = \left[\frac{x^2}{2} + \frac{x}{2} \right]_0^1 = 1 \end{aligned}$$

(ii) General region between $y = g_1(x)$ and $y = g_2(x)$, $(g_1(x) \leq g_2(x), a \leq x \leq b)$

$$\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

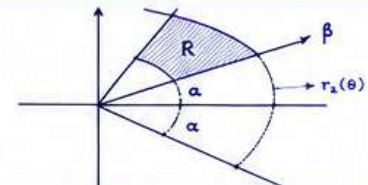


Example : Evaluate $I = \iint_R xy dA$, where R is bounded by $y = 0, y = x, x = 0, x = 1$.

$$\begin{aligned} \text{Sol : } I &= \int_0^1 \int_0^x xy dy dx = \int_0^1 x \left[\frac{y^2}{2} \right]_0^x dx \\ &= \int_0^1 \frac{x^3}{2} dx = \left[\frac{x^4}{8} \right]_0^1 = \frac{1}{8} \end{aligned}$$

(iii) Polar coordinates ($R : a \leq \theta \leq \beta, r_1(\theta) \leq r \leq r_2(\theta)$)

$$\iint_R f(x, y) dA = \int_a^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$



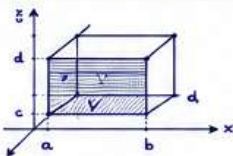
Example : Evaluate $\iint_R r dA$ over R given by $0 \leq r \leq 2, 0 \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned} \text{Sol : } I &= \int_0^{\pi/2} \int_0^2 r \cdot r dr d\theta = \int_0^{\pi/2} \int_0^2 r^2 dr d\theta \\ &= \int_0^{\pi/2} \left[\frac{r^3}{3} \right]_0^2 d\theta = \int_0^{\pi/2} \frac{8}{3} d\theta = \frac{8}{3} \cdot \frac{\pi}{2} = \frac{4\pi}{3} \end{aligned}$$

5. TRIPLE INTEGRALS

(i) Rectangular box $[a, b] \times [c, d] \times [e, f]$

$$\iiint_V f(x, y, z) dV = \int_a^b \int_c^d \int_e^f f(x, y, z) dz dy dx$$



(ii) General order of integration

$$\iiint_V f(x, y, z) dV = \int \int \int f(x, y, z) dz dy dx$$

(choose order as per limits)

Example : Evaluate $\iiint_V 1 dV$ over the cube

$0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

$$\begin{aligned} \text{Sol : } \int_0^1 \int_0^1 \int_0^1 1 dz dy dx &= \int_0^1 \int_0^1 [z]_0^1 dy dx \\ &= \int_0^1 \int_0^1 1 dy dx = \int_0^1 [y]_0^1 dx = \int_0^1 1 dx = 1 \end{aligned}$$

(Volume of unit cube = 1)

(iii) In cylindrical coordinates

$$\iiint_V f(r, \theta, z) dV = \int_a^\beta \int_{r_1}^{r_2} \int_{z_1}^{z_2} f(r, \theta, z) r dz dr d\theta$$

Example : Volume of cylinder $x^2 + y^2 \leq a^2, 0 \leq z \leq h$

$$\begin{aligned} \text{Sol : } V &= \int_0^{2\pi} \int_0^a \int_0^h r dz dr d\theta = \int_0^{2\pi} \int_0^a [rz]_0^h dr d\theta \\ &= \int_0^{2\pi} \int_0^a hr dr d\theta = \int_0^{2\pi} h \left[\frac{r^2}{2} \right]_0^a d\theta \\ &= \int_0^{2\pi} \frac{ha^2}{2} d\theta = \frac{ha^2}{2} [\theta]_0^{2\pi} = \pi a^2 h \end{aligned}$$

6. GATE LEVEL PROBLEMS

(i) Find the extrema of $z = x^2 + y^2 - 3xy$.

- (ii) Min at $(1, 1)$
(B) Max at $(1, -1)$
(C) Min at $(-1, 1)$
(D) Max at $(-1, -1)$

Sol :

$$\begin{aligned} f_x &= 2x - 3y, f_y = 2y - 3x \\ f_x = f_y = 0 &\Rightarrow x^2 - y^2 = 0, y^2 - x^2 = 0 \\ &\Rightarrow (x, y) = (0, 0), (1, 1), (-1, -1) \\ \text{At } (1, 1): f_{xx} &= 2, f_{yy} = 2, f_{xy} = -3 \\ D &= 2 \cdot 2 - (-3)^2 = -5 < 0, f_{xx} > 0 \\ &\Rightarrow \text{Min at } (1, 1). \text{ Ans : (A)} \end{aligned}$$

(ii) Extremum of $f(x, y) = x^2 + y^2$ subject to $x + 2y = 4$ is

- (A) Max = $\frac{16}{5}$ (B) Min = $\frac{16}{5}$
(C) Max = $\frac{32}{5}$ (D) Min = 0

Sol :

$$\begin{aligned} 2x &= \lambda, 2y = 2\lambda, x + 2y = 4 \\ &\Rightarrow x = \frac{\lambda}{2}, y = \lambda \\ \frac{\lambda}{2} + 2\lambda &= 4 \Rightarrow \lambda = \frac{8}{5} \\ &\Rightarrow (x, y) = \left(\frac{4}{5}, \frac{8}{5} \right) \\ f_{\min} &= \frac{16}{25} + \frac{64}{25} = \frac{80}{25} = \frac{16}{5} \text{ Ans : (B)} \end{aligned}$$

(iii) $\iint_R (x+y) dA$ over $R = [0, 1] \times [0, 2]$.

- (A) $\frac{5}{2}$ (B) 3
(C) 4 (D) $\frac{7}{2}$

$$\begin{aligned} \text{Sol : } \int_0^1 \int_0^2 (x+y) dy dx &= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^2 dx \\ &= \int_0^1 (2x + 2) dx = [x^2 + 2x]_0^1 = 1 + 2 = 3 \end{aligned}$$

Ans : (B)

(iv) $\iiint_V 1 dV$ over $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2$

- (A) 2π (B) π
(C) 4π (D) 2

$$\begin{aligned} \text{Sol : } V &= \int_0^{2\pi} \int_0^1 \int_0^2 r dz dr d\theta = \int_0^{2\pi} \int_0^1 [rz]_0^2 dr d\theta \\ &= \int_0^{2\pi} 2 \left[\frac{r^2}{2} \right]_0^1 d\theta = \int_0^{2\pi} 1 d\theta = 2\pi \end{aligned}$$

Ans : (A)

7. SHORT TRICKS

- ✓ Critical points come from $f_x = 0, f_y = 0$.
- ✓ Use $D = f_{xx}f_{yy} - (f_{xy})^2$ for quick test.
- ✓ For constraint use $\nabla f = \lambda \nabla g$.
- ✓ For double integrals, choose order of integration to match limits.
- ✓ In polar/cylindrical, don't forget the Jacobian: $dA = r dr d\theta, dV = r dr d\theta dz$.

8. COMMON GATE MISTAKES

- ✗ Forgetting second derivative test conditions.
- ✗ Not checking all critical points.
- ✗ Ignoring boundaries in constrained problems.
- ✗ Wrong order/limits in multiple integrals.
- ✗ Missing 'r' in polar/cylindrical integrals.
- ✗ Sign mistakes in partial derivatives.

9. QUICK REFERENCE FORMULAS

- $D = f_{xx}f_{yy} - (f_{xy})^2$
- Constraint: $\nabla f = \lambda \nabla g$
- Double integral: $\iint_R f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$
- Polar: $dA = r dr d\theta$
- Triple (cyl.): $dV = r dr d\theta dz$
- Volume of cylinder: $\pi a^2 h$

10. KEY REMINDERS

- ★ Classify with D, not just f_{xx} .
- ★ Check boundary in constrained extrema.
- ★ Sketch region before integrating.
- ★ Simplify before integrating.
- ★ Practice GATE PYQs - very important!



FOURIER SERIES

Fourier Coefficients, Even/Odd Functions,
Half-Range Series and Important Results

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1. INTRODUCTION

A function $f(x)$ of period $2L$ (i.e., $f(x+2L)=f(x)$) which satisfies Dirichlet conditions, can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

where a_0, a_n, b_n are called Fourier coefficients.

2. FOURIER COEFFICIENTS

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad n \geq 1$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx, \quad n \geq 1$$

$n = 1, 2, 3, \dots$

3. EVEN AND ODD FUNCTIONS

(i) Even function: $f(-x) = f(x)$

$$\Rightarrow b_n = 0$$

$$a_0 = \frac{2}{L} \int_0^L f(x) dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

Hence,

If $f(x)$ is even,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

(ii) Odd function: $f(-x) = -f(x)$

$$\Rightarrow a_0 = 0, \quad a_n = 0$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

If $f(x)$ is odd,

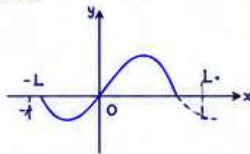
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

4. HALF-RANGE FOURIER SERIES

(i) Sine series (odd extension): $0 < x < L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$\text{where } b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

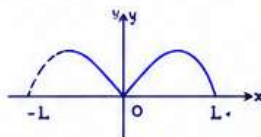


(ii) Cosine series (even extension): $0 < x < L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$\text{where } a_0 = \frac{2}{L} \int_0^L f(x) dx,$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$



5. IMPORTANT RESULTS

$$\star \int_{-L}^L \cos \frac{n\pi x}{L} dx = \begin{cases} 2L, & n = 0 \\ 0, & n \geq 1 \end{cases}$$

$$\star \int_{-L}^L \sin \frac{n\pi x}{L} dx = 0, \quad \forall n$$

$$\star \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \begin{cases} L, & m = n \neq 0 \\ 0, & m \neq n \end{cases}$$

$$\star \int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} L, & m = n \\ 0, & m \neq n \end{cases}$$

$$\star \int_{-L}^L \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = 0, \quad \forall m, n$$

At points of discontinuity, the series converges to $\frac{f(x^+) + f(x^-)}{2}$ (average of limits).

6. IMPORTANT SPECIAL CASES

(i) $f(x) = 1$

$$a_0 = 2, \quad a_n = 0, \quad b_n = 0$$

$$f(x) = 1$$

(ii) $f(x) = x$ ($-L < x < L$) (odd)

$$b_n = \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} dx = \frac{2L(-1)^{n+1}}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin \frac{n\pi x}{L}$$

(iii) $f(x) = x^2$ ($-L < x < L$) (even)

$$a_0 = \frac{2L^2}{3}, \quad a_n = \frac{4L^2(-1)^n}{n^2\pi^2}, \quad b_n = 0$$

7. GATE LEVEL PROBLEMS

(i) Find Fourier series of $f(x) = x$ in $(-\pi, \pi)$.

Sol:

$f(x)$ is odd $\Rightarrow a_0 = 0, a_n = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$$

$$= \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

Ans:

(ii) Find Fourier series of $f(x) = x^2$ in $(-\pi, \pi)$.

Sol: $f(x)$ is even $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx$$

$$= \frac{2(-1)^n}{n^2}$$

$$f(x) = \frac{\pi^2}{3} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

Ans:

(iii) For $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & -\pi < x < 0 \end{cases}$

find Fourier series in $(-\pi, \pi)$.

Sol:

$$a_n = \frac{1}{\pi} \int_0^{\pi} 1 dx = 1$$

$$a_n = \frac{1}{\pi} \int_0^{\pi} \cos nx dx = 0$$

$$b_n = \frac{1}{\pi} \int_0^{\pi} \sin nx dx = \frac{1-(-1)^n}{n\pi}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin nx$$

Ans:

(iv) Find Fourier sine series of $f(x) = x$ in $(0, \pi)$.

$$\text{Sol: } b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx = \frac{2(-1)^{n+1}}{n}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx, \quad 0 < x < \pi$$

Ans:

8. SHORT TRICKS

- ✓ Check even/odd nature first to reduce work.
- ✓ For half-range series, extend the function evenly (cosine) or oddly (sine).
- ✓ Use orthogonality results to evaluate coefficients quickly.
- ✓ Always write the interval of definition clearly.

9. COMMON GATE MISTAKES

- ✗ Forgetting the factor $\frac{1}{L}$ in coefficients.
- ✗ Wrong limits in half-range integrals.
- ✗ Ignoring symmetry and calculating a_n, b_n unnecessarily.
- ✗ Wrong signs while integrating by parts.
- ✗ Not simplifying final series.

10. QUICK REFERENCE FORMULAS

$$\star \text{ General FS (period } 2L): f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

★ Even function: $b_n = 0$ Odd function: $a_0 = a_n = 0$

★ Half-range (sine): $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$

★ Half-range (cosine): $a_0 = \frac{2}{L} \int_0^L f(x) dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$



VECTOR CALCULUS - BASICS

Vector Identities, Gradient, Directional Derivative,
Divergence and Curl

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1. SCALAR AND VECTOR FIELDS

A scalar field in $\mathbb{R}^3$ is a function $\phi(x,y,z)$ that assigns a scalar to each point.

A vector field in $\mathbb{R}^3$ $\vec{F}(x,y,z) = P\hat{i} + Q\hat{j} + R\hat{k}$ where P, Q, R are scalar functions.

2. VECTOR IDENTITIES

- (i) $\nabla \times (\nabla \phi) = \vec{0}$
- (ii) $\nabla \cdot (\nabla \times \vec{A}) = 0$
- (iii) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$
- (iv) $\nabla \cdot (\phi \vec{A}) = \phi(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla \phi$
- (v) $\nabla \times (\phi \vec{A}) = \phi(\nabla \times \vec{A}) + \nabla \phi \times \vec{A}$
- (vi) $\nabla(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A})$
- (vii) $\nabla^2 \phi = \nabla \cdot (\nabla \phi)$ (Laplacian of ϕ)
 $\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times (\nabla \times \vec{A})$

3. GRADIENT (grad ϕ)

For a scalar field $\phi(x,y,z)$,

$$\nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

- Points in the direction of maximum increase of ϕ .
- Magnitude $|\nabla \phi|$ is the maximum rate of increase.

Example: $\phi(x,y,z) = x^2y + yz^2$

$$\begin{aligned} \nabla \phi &= \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} \\ &= 2xy \hat{i} + (x^2 + z^2) \hat{j} + 2yz \hat{k} \end{aligned}$$

4. DIRECTIONAL DERIVATIVE

Rate of change of ϕ at a point P in the direction of a unit vector $\hat{u}$.

$$D_{\hat{u}} \phi = \nabla \phi \cdot \hat{u}$$

- If $\hat{u} = \hat{i}, \hat{j}, \hat{k}$ gives $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}$

Example: $\phi = x^2y + yz^2$, direction $\hat{u} = \frac{\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{6}}$

$$\begin{aligned} \text{At } (1,1,1), \quad \nabla \phi &= 2\hat{i} + 2\hat{j} + 2\hat{k} \\ D_{\hat{u}} \phi &= (2\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{6}} \\ &= \frac{2+4-2}{\sqrt{6}} = \frac{4}{\sqrt{6}} \end{aligned}$$

5. DIVERGENCE (div $\vec{F}$)

For $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$,

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

- Scalar quantity.
- Physically: measures net outward flux per unit volume at a point.
 $\text{div } \vec{F} > 0 \rightarrow \text{source}$
 $\text{div } \vec{F} < 0 \rightarrow \text{sink}$
 $\text{div } \vec{F} = 0 \rightarrow \text{solenoidal field}$

Example: $\vec{F} = x^2\hat{i} + xy\hat{j} + z\hat{k}$

$$\begin{aligned} \text{div } \vec{F} &= \frac{\partial(x^2)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial(z)}{\partial z} \\ &= 2x + x + 1 = 3x + 1 \end{aligned}$$

6. CURL (rot $\vec{F}$)

For $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$

- Vector quantity.
- Direction: given by right hand thumb rule.
- Magnitude: measures rotation (circulation density).

Example: $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$

$$\begin{aligned} \nabla \times \vec{F} &= \left(\frac{\partial(zx)}{\partial y} - \frac{\partial(yz)}{\partial z} \right) \hat{i} + \left(\frac{\partial(xy)}{\partial z} - \frac{\partial(zx)}{\partial x} \right) \hat{j} \\ &\quad + \left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y} \right) \hat{k} \\ &= (-z)\hat{i} + (-z)\hat{j} + (-y)\hat{k} = -z\hat{i} - z\hat{j} - y\hat{k} \end{aligned}$$

7. OPERATOR FORM IN CARTESIAN COORDINATES

Del operator:

$$\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

Hence,

$$\text{grad } \phi = \nabla \phi$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$\text{Laplacian } \nabla^2 \phi = \nabla \cdot (\nabla \phi)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

8. GATE LEVEL PROBLEMS

- (i) If $\phi = xyz$, find $\nabla \phi$ and $D_{\hat{u}} \phi$ at $(1,1,1)$ in the direction $\hat{u} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$.

Sol: $\nabla \phi = yz\hat{i} + xz\hat{j} + xy\hat{k}$

At $(1,1,1)$, $\nabla \phi = \hat{i} + \hat{j} + \hat{k}$

$$\begin{aligned} D_{\hat{u}} \phi &= (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} \\ &= \frac{3}{\sqrt{3}} = \sqrt{3} \end{aligned}$$

Ans: $\sqrt{3}$

- (ii) If $\vec{F} = x^2\hat{i} + xy\hat{j} + z^2\hat{k}$, find $\text{div } \vec{F}$.

Sol: $\text{div } \vec{F} = \frac{\partial(x^2)}{\partial x} + \frac{\partial(xy)}{\partial y} + \frac{\partial(z^2)}{\partial z}$

$$= 2x + x + 2z = 3x + 2z$$

Ans: $3x + 2z$

- (iii) If $\vec{F} = xy\hat{i} + yz\hat{j} + zx\hat{k}$, find $\nabla \times \vec{F}$.

Sol: $\nabla \times \vec{F} = -z\hat{i} - z\hat{j} - y\hat{k}$

(from Ex. above)

Ans: $-z\hat{i} - z\hat{j} - y\hat{k}$

- (iv) If $\vec{F} = \nabla \phi$ for some scalar ϕ , what is $\nabla \times \vec{F}$?

- (A) 0 (B) $\nabla \phi$
(C) $\vec{F}$ (D) Not defined

Sol: Since $\vec{F} = \nabla \phi$,
 $\nabla \times \vec{F} = \nabla \times (\nabla \phi) = \vec{0}$ (identity)

Ans: (A)

9. SHORT TRICKS

- ✓ Remember determinant method for curl quickly.
- ✓ For gradient, write all partials in the same order $\hat{i}, \hat{j}, \hat{k}$.
- ✓ For divergence, just add diagonal terms.
- ✓ For directional derivative, take dot product with unit vector.

10. COMMON GATE MISTAKES

- ✗ Forgetting unit vector in directional derivative.
- ✗ Wrong order of partial derivatives in curl.
- ✗ Sign mistakes in vector identities.
- ✗ Confusing gradient ($\nabla \phi$) with curl ($\nabla \times \vec{F}$).

11. QUICK REFERENCE FORMULAS

- $\text{grad } \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$
- $D_{\hat{u}} \phi = \nabla \phi \cdot \hat{u}$
- $\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
- $\text{curl } \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$

12. KEY REMINDERS

- ★ Gradient $\rightarrow$ maximum increase
- ★ Divergence $\rightarrow$ source/sink
- ★ Curl $\rightarrow$ rotation
- ★ Always check units and signs.
- ★ Practice GATE PYQs on vector calculus!



LINE, SURFACE & VOLUME INTEGRALS

Evaluation Techniques, Vector-Field Integrals and Applications

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1. LINE INTEGRALS (SCALAR & VECTOR)

(i) Scalar line integral

If f is a scalar function and C is a smooth curve from A to B , then

$$\int_C f \, ds = \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt$$

(ii) Vector line integral

If $\vec{F}$ is a vector field,

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

where $\vec{r}(t)$ is the vector equation of C .

Example:

Evaluate $\int_C (x \, dx + y \, dy)$ along the line from $(0,0)$ to $(1,1)$.

Sol:

Parametrize $\vec{r}(t) = \langle t, t \rangle$, $0 \leq t \leq 1$

$$d\vec{r} = \langle 1, 1 \rangle dt$$

$$\int_C (x \, dx + y \, dy) = \int_0^1 (t \, dt + t \, dt) = \int_0^1 2t \, dt = 1$$

4. VOLUME INTEGRALS

(i) Scalar volume integral

$$\iiint_V f(x, y, z) \, dV$$

(ii) Vector volume integral

$$\iiint_V \vec{F}(x, y, z) \, dV$$

Example:

Evaluate $\iiint_V (x+y+z) \, dV$, where V is the cube $0 \leq x, y, z \leq 1$.

Sol:

$$\begin{aligned} \iiint_V (x+y+z) \, dV &= \int_0^1 \int_0^1 \int_0^1 (x+y+z) \, dz \, dy \, dx \\ &= \int_0^1 \int_0^1 \left[xz + yz + \frac{z^2}{2} \right]_0^1 \, dy \, dx \\ &= \int_0^1 \int_0^1 \left(x + y + \frac{1}{2} \right) \, dy \, dx \\ &= \int_0^1 \left[xy + \frac{y^2}{2} + \frac{1}{2}y \right]_0^1 \, dx \\ &= \int_0^1 \left(x + \frac{1}{2} + \frac{1}{2} \right) \, dx = \int_0^1 (x+1) \, dx = \frac{3}{2} \end{aligned}$$

7. DIVERGENCE THEOREM (GAUSS)

If S is a closed surface enclosing volume V outward oriented, and $\vec{F}$ has continuous partial derivatives in V , then

$$\iiint_V \vec{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$$

Example:

Evaluate $\iiint_V (\hat{x} + \hat{y} + \hat{z}) \cdot \hat{n} \, dS$ over the surface of cube $0 \leq x, y, z \leq 1$.

Sol:

$$\nabla \cdot \vec{F} = 1 + 1 + 1 = 3$$

$$\Rightarrow \text{Flux} = \iiint_V 3 \, dV = 3 \times (\text{Vol. of cube}) = 3$$

10. GATE LEVEL PROBLEMS

(i) Evaluate $\int_C (2x \, dy - y \, dx)$, where C is the circle $x^2 + y^2 = a^2$ (positively oriented).

- (A) 0 (B) πa^2
(C) $2\pi a^2$ (D) $4\pi a^2$

Sol: By Green's theorem, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - (-1) = 3$
Value = $3 \times \pi a^2 = 3\pi a^2$

Ans: None of the above (Value = $3\pi a^2$)

(ii) Flux of $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ outward through the cube $0 \leq x, y, z \leq 1$ is

- (A) 0 (B) 1
(C) 2 (D) 3

Sol: $\nabla \cdot \vec{F} = 0 + 0 + 0 = 0$

$$\text{Flux} = 0$$

Ans: (A)

(iii) Evaluate $\int_C (x \, dy - y \, dx)$, where C is the boundary of the square with vertices $(0,0), (a,0), (a,a), (0,a)$.

- (A) 0 (B) a^2
(C) $2a^2$ (D) $4a^2$

Sol: By Green's theorem, value = $\iint_R 2 \, dA$
= $2 \times (\text{Area of square}) = 2a^2$

Ans: (C)

(iv) If $\vec{F} = -y\hat{i} + x\hat{j} + 0\hat{k}$, find $\int_C \vec{F} \cdot d\vec{r}$, where C is the circle $x^2 + y^2 = a^2$ in xy -plane.

- (A) 0 (B) πa^2
(C) $2\pi a^2$ (D) $4\pi a^2$

Sol: Using Stokes, value = $2\pi a^2$

Ans: (C)

2. INDEPENDENCE OF PATH (CONSERVATIVE FIELDS)

If $\vec{F}$ is conservative (i.e., $\vec{F} = \nabla\phi$ for some ϕ), then for any curve C from A to B ,

$$\int_C \vec{F} \cdot d\vec{r} = \phi(B) - \phi(A)$$

If $\nabla \times \vec{F} = \vec{0}$ in a simply connected region $\Rightarrow \vec{F}$ is conservative.

Example:

Evaluate $\int_C \nabla(x^2y + yz) \cdot d\vec{r}$ from $(0,0,0)$ to $(1,2,3)$ along any path.

Sol:

Here, $\phi = x^2y + yz$

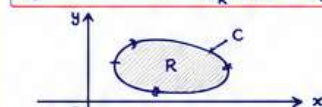
$$\text{Value} = \phi(1,2,3) - \phi(0,0,0)$$

$$= (1)^2(2) + (2)(3) - 0 = 8$$

5. GREEN'S THEOREM (PLANE)

If C is positively oriented simple closed curve bounding region R in xy -plane, and $\vec{F} = P\hat{i} + Q\hat{j}$, where P, Q have continuous partial derivatives in R , then

$$\oint_C P \, dx + Q \, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$



Example:

Evaluate $\oint_C (x \, dy - y \, dx)$, $C: x^2 + y^2 = a^2$.

Sol:

Here, $P = -y$, $Q = x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

$$= \iint_R 2 \, dA = 2 \times \text{Area of circle} = 2 \times \pi a^2$$

3. SURFACE INTEGRALS

(i) Scalar surface integral

If f is scalar and S is a surface,

$$\iint_S f \, dS$$

(ii) Vector surface integral (flux)

If $\vec{F}$ is vector field,

$$\iint_S \vec{F} \cdot \hat{n} \, dS$$

where $\hat{n}$ is unit normal to S .

Example:

Evaluate the flux of $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ outward through the sphere $x^2 + y^2 + z^2 = a^2$.

Sol:

By Divergence theorem (see section 5):

$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V (\nabla \cdot \vec{F}) \, dV$$

$$\nabla \cdot \vec{F} = 1 + 1 + 1 = 3$$

$$= \iiint_V 3 \, dV = 3 \times \text{Vol. of sphere} = 3 \times \frac{4}{3} \pi a^3 = 4\pi a^3$$

6. STOKES' THEOREM

If S is an oriented smooth surface with boundary C (positively oriented), and $\vec{F}$ has continuous partial derivatives, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, dS$$

Example:

Verify Stokes' theorem for $\vec{F} = -y\hat{i} + x\hat{j}$, $C: x^2 + y^2 = a^2$ in xy -plane.

Sol:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \hat{k}(1+1) = 2\hat{k}$$

$$\text{RHS} = \iint_S (2\hat{k}) \cdot \hat{k} \, dS = 2 \times \pi a^2$$

$$\text{LHS} = \oint_C (-y \, dx + x \, dy) \text{ (anticlockwise)} = 2\pi a^2 \quad \checkmark$$

9. APPLICATIONS

- Work done by a force along a path:

$$W = \int_C \vec{F} \cdot d\vec{r}$$

- Fluid flow (flux) across a surface:

$$\iint_S \vec{v} \cdot \hat{n} \, dS$$

- Circulation around a boundary:

$$\oint_C \vec{F} \cdot d\vec{r}$$

- Total flux leaving a closed surface:

$$\iint_S \vec{F} \cdot \hat{n} \, dS$$

- Mass/charge inside a volume from density ρ :

$$\iiint_V \rho \, dV$$

11. SHORT TRICKS

- ✓ For conservative fields, evaluate using potential difference.
- ✓ For flux through closed surfaces, use divergence theorem.
- ✓ For circulation and planar regions, use Green's theorem.
- ✓ Always check orientation of C and S before applying theorems.

12. COMMON MISTAKES

- ✗ Forgetting orientation (normal or curve).
- ✗ Missing minus sign in $d\vec{r}$ or $d\vec{S}$.
- ✗ Wrong Jacobian in change of variables.
- ✗ Applying theorems without checking conditions.

13. QUICK REFERENCE FORMULAS

- $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$
- $\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
- $\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$
- $\int_C \vec{F} \cdot d\vec{r} = \int_a^b (\nabla \cdot \vec{F}) \cdot \hat{n} \, dS$
- $\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$

14. KEY REMINDERS

- ★ Practice varied examples.
- ★ Draw neat diagrams.
- ★ Check limits and orientation.
- ★ Understand the meaning (work, flux, circulation).
- ★ Theorems save time in GATE problems!



VECTOR INTEGRAL THEOREMS

Green's Theorem, Stokes' Theorem,
Gauss (Divergence) Theorem + GATE Problems

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1. GREEN'S THEOREM (PLANE)

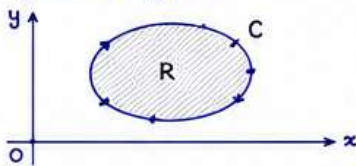
If C is a positively oriented, simple, closed curve bounding a region R in the xy -plane, and $P(x,y)$, $Q(x,y)$ have continuous partial derivatives on an open region containing R , then

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Where

C : Boundary of R (positively oriented)

R : Region in the xy -plane



EXAMPLE (Green's Theorem)

Evaluate $\oint_C (x^2 - y) dx + (xy + y^2) dy$, where C is the positively oriented boundary of the square: $0 \leq x \leq a$, $0 \leq y \leq a$.

Sol:

Here, $P = x^2 - y$, $Q = xy + y^2$

$$\frac{\partial Q}{\partial x} = y, \quad \frac{\partial P}{\partial y} = -1$$

$$\rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y - (-1) = y + 1$$

$$\begin{aligned} \oint_C P dx + Q dy &= \iint_R (y+1) dA \\ &= \int_0^a \int_0^a (y+1) dy dx \\ &= \int_0^a \left[\frac{y^2}{2} + y \right]_0^a dx \\ &= \int_0^a \left(\frac{a^2}{2} + a \right) dx = a \left(\frac{a^2}{2} + a \right) \\ &= \frac{a^3}{2} + a^2 \end{aligned}$$

Ans: $\frac{a^3}{2} + a^2$

4. GATE LEVEL PROBLEMS

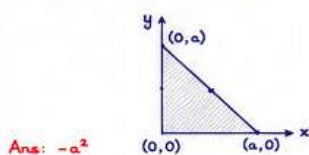
(i) Evaluate $\oint_C (y dx - x dy)$ where C is the boundary of the triangle with vertices $(0,0)$, $(a,0)$, $(0,a)$ taken in positive (anticlockwise) sense.

Sol: By Green's theorem, $P = y$, $Q = -x$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1 - 1 = -2$$

$$\text{Area of triangle} = \frac{1}{2} a^2$$

$$\Rightarrow \oint_C (y dx - x dy) = -2 \times \frac{1}{2} a^2 = -a^2$$



Ans: $-a^2$

(ii) Evaluate $\oint_C (y dx + x dy + z dz)$, where C is the boundary of the portion of the plane $x + y + z = a$ in the first octant, taken positively.

Sol: By Stokes' theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS \text{ where } \vec{F} = y\hat{i} + x\hat{j} + z\hat{k}$$

$$\begin{aligned} \nabla \times \vec{F} &= \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(1-0) = -\hat{i} + \hat{j} + \hat{k} \\ \text{For plane } x+y+z=a \text{ in first octant, projected region in } xy\text{-plane is triangle with area } \frac{1}{2} a^2 \\ \text{Unit normal (upward direction)} &= \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) dS_y \end{aligned}$$

$$\begin{aligned} \text{Hence } (\nabla \times \vec{F}) \cdot \hat{n} &= (-\hat{i} + \hat{j} + \hat{k}) \cdot \frac{1}{\sqrt{3}} (\hat{i} + \hat{j} + \hat{k}) = \frac{1}{\sqrt{3}} \\ \Rightarrow \oint_C \vec{F} \cdot d\vec{r} &= \iint_S \frac{1}{\sqrt{3}} dS = \frac{1}{\sqrt{3}} \times \frac{1}{2} a^2 \end{aligned}$$

Ans: $\frac{a^2}{2\sqrt{3}}$

2. STOKES' THEOREM (SURFACE)

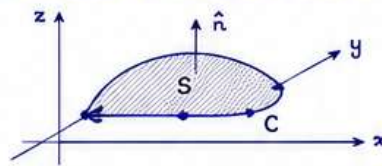
If S is a smooth oriented surface with positively oriented boundary C , and $\vec{F}$ has continuous partial derivatives on a region containing S , then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$$

Where

C : Boundary of S (right hand rule)

$\hat{n}$: Unit normal to S consistent with C



EXAMPLE (Stokes' Theorem)

Evaluate $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and C is the boundary of the surface $S: z = x^2 + y^2$ for $z \leq 1$ (upward orientation).

Sol:

By Stokes' theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$$

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} \\ &= \hat{i}(0-1) - \hat{j}(0-1) + \hat{k}(1-0) \\ &= -\hat{i} + \hat{j} + \hat{k} \end{aligned}$$

For $z = x^2 + y^2$, upward $\hat{n} dS = -f_x \hat{i} - f_y \hat{j} + \hat{k} dA$ where $f_x = 2x$, $f_y = 2y$.

So, $\hat{n} dS = -2x\hat{i} - 2y\hat{j} + \hat{k} dA$

$$\begin{aligned} \text{Hence, } (\nabla \times \vec{F}) \cdot \hat{n} dS &= (-\hat{i} + \hat{j} + \hat{k}) \cdot (-2x\hat{i} - 2y\hat{j} + \hat{k}) dA \\ &= (2x - 2y + 1) dA \end{aligned}$$

Projection on xy -plane: $x^2 + y^2 \leq 1$ (circle)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{x^2+y^2 \leq 1} (2x - 2y + 1) dA$$

The integrals of $2x$ and $-2y$ over the circle are zero (by symmetry). Hence,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{x^2+y^2 \leq 1} 1 dA = \text{Area of circle} = \pi$$

Ans: π

3. GAUSS (DIVERGENCE) THEOREM (VOLUME)

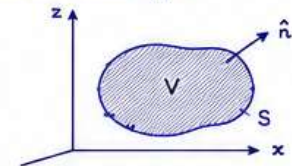
If V is a volume in $\mathbb{R}^3$ with piecewise smooth closed surface S (outward orientation), and $\vec{F}$ has continuous partial derivatives on a region containing V , then

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V (\nabla \cdot \vec{F}) dV$$

Where

S : Closed surface bounding V (outward normal)

V : Volume enclosed by S .



EXAMPLE (Gauss Theorem)

Evaluate $\iint_S \vec{F} \cdot \hat{n} dS$, where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the closed surface of the cube $0 \leq x, y, z \leq a$.

Sol:

By Gauss theorem,

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V (\nabla \cdot \vec{F}) dV$$

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\text{Hence, } \iiint_V 3 dV = 3 \times \text{Volume of cube} = 3a^3$$

Ans: $3a^3$

(iii) Evaluate $\iint_S (x dy dz + y dz dx + z dx dy)$, where S is the closed surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Sol: Note that $x dy dz + y dz dx + z dx dy$

$$= \vec{F} \cdot \hat{n} dS \text{ for } \vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

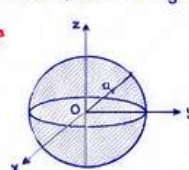
By Gauss theorem,

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V (\nabla \cdot \vec{F}) dV$$

$$\nabla \cdot \vec{F} = 3$$

$$= 3 \times \text{Volume of sphere} = 3 \times \frac{4}{3} \pi a^3 = 4\pi a^3$$

Ans: $4\pi a^3$



(iv) If $\vec{F} = 2y\hat{i} + 2x\hat{j} + 2z\hat{k}$, evaluate $\iint_S \vec{F} \cdot \hat{n} dS$ where S is the closed surface of the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$ (outward orientation).

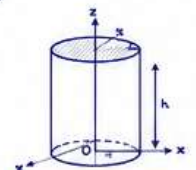
Sol: By Gauss theorem,

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V (\nabla \cdot \vec{F}) dV$$

$$\nabla \cdot \vec{F} = \frac{\partial (2y)}{\partial x} + \frac{\partial (2x)}{\partial y} + \frac{\partial (2z)}{\partial z} = 0 + 0 + 0 = 0$$

Hence, flux = 0

Ans: 0



5. SHORT TRICKS

- ✓ Always check orientation before using theorems.
- ✓ For Green's theorem, sketch the region and find $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ first.
- ✓ For Stokes, use right hand rule for direction (Curl $\rightarrow$ Boundary).
- ✓ For Gauss, compute $\nabla \cdot \vec{F}$ first, then integrate over volume (much easier!).

6. COMMON MISTAKES

- ✗ Wrong orientation (clockwise vs anticlockwise).
- ✗ Forgetting unit normal direction.
- ✗ Missing minus signs in Curl.
- ✗ Using Green's theorem in 3D.
- ✗ Not checking smoothness/continuity conditions.

7. QUICK REFERENCE FORMULAS

- Green: $\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$
- Stokes: $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} dS$
- Gauss: $\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V (\nabla \cdot \vec{F}) dV$
- $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
- $\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

8. KEY REMINDERS

- ★ Understand meaning, not just formulas.
- ★ Practice varied GATE level problems.
- ★ Draw neat diagrams and check orientation.
- ★ Theorems save time in tough integrals!

FIRST-ORDER DIFFERENTIAL EQUATIONS

Linear & Nonlinear Equations, Exact Equations,
Bernoulli Equations and Applications

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1. FIRST-ORDER DIFFERENTIAL EQUATION

An equation involving $y(x)$ and its first derivative

$\frac{dy}{dx}$ is called

a first-order differential equation.

General form:

$$\frac{dy}{dx} = F(x, y)$$

or

$$M(x, y)dx + N(x, y)dy = 0$$

2. LINEAR FIRST-ORDER DIFFERENTIAL EQUATION

Equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Solution:

Integrating factor (I.F.)

$$I.F. = e^{\int P(x) dx}$$

General solution:

$$y e^{\int P(x) dx} = \int Q(x) e^{\int P(x) dx} dx + C$$

Example: Solve $\frac{dy}{dx} + y = e^x$

Sol: I.F. = $e^{\int 1 dx} = e^x$

$$y e^x = \int e^x \cdot e^x dx + C = \int e^{2x} dx + C = \frac{1}{2} e^{2x} + C$$

$$\Rightarrow y = \frac{1}{2} e^x + C e^{-x}$$

3. NONLINEAR FIRST-ORDER DIFFERENTIAL EQUATIONS

(a) Separable Equations

If $\frac{dy}{dx} = X(x)Y(y)$, then

$$\frac{dy}{Y(y)} = X(x) dx$$

Example:

$$\frac{dy}{dx} = xy$$

Sol: $\frac{dy}{y} = x dx \Rightarrow \ln|y| = \frac{x^2}{2} + C$

$$\Rightarrow y = C e^{x^2/2}$$

(b) Homogeneous Equations

If $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, put $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

4. EXACT DIFFERENTIAL EQUATIONS

Equation of the form $M(x, y)dx + N(x, y)dy = 0$

is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution:

There exists a function $\phi(x, y)$ such that

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = M dx + N dy$$

$$\Rightarrow \phi(x, y) = C$$

Example:

Solve $(2xy + 3)dx + (x^2 + 4y)dy = 0$

Sol: $M = 2xy + 3$, $N = x^2 + 4y$

$$\frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = 2x \quad (\text{Exact})$$

Let $\phi_x = 2xy + 3$

$$\phi = x^2 y + 3x + g(y)$$

$$\phi_y = x^2 + g'(y) = x^2 + 4y \Rightarrow g'(y) = 4y$$

$$g(y) = 2y^2$$

$$\Rightarrow \phi = x^2 y + 3x + 2y^2 = C$$

5. LINEAR EQUATION (ANOTHER FORM)

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\text{Homogeneous part: } \frac{dy}{dx} + P(x)y = 0$$

Particular integral by method of I.F.

Example:

Solve $\frac{dy}{dx} + \frac{2}{x}y = x^2$

Sol:

$$I.F. = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

$$y x^2 = \int x^2 \cdot x^2 dx + C = \int x^4 dx + C = \frac{x^5}{5} + C$$

$$\Rightarrow y = \frac{x^3}{5} + \frac{C}{x^2}$$

6. BERNOULLI'S EQUATION

Equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \quad n \neq 0, 1$$

Solution:

Divide by y^n and put $v = y^{1-n}$

Then $\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$ (Linear in v)

Example:

Solve $\frac{dy}{dx} + \frac{y}{x} = xy^2$

Sol: Here $n = 2$, Divide by y^2 ,

$$y^{-2} \frac{dy}{dx} + \frac{1}{x} y^{-1} = x$$

Put $v = y^{-1} \Rightarrow \frac{dv}{dx} = -y^{-2} \frac{dy}{dx}$

$$-\frac{dv}{dx} + \frac{1}{x} v = x \Rightarrow \frac{dv}{dx} - \frac{1}{x} v = -x$$

$$I.F. = e^{\int -\frac{1}{x} dx} = x^{-1}$$

$$v x^{-1} = \int -x \cdot x^{-1} dx + C = -\frac{x^2}{2} + C$$

$$v = -\frac{x^3}{2} + Cx \Rightarrow y = \frac{1}{v} = \frac{1}{Cx - \frac{x^3}{2}}$$

7. APPLICATIONS

(i) Population growth (no restriction)

$$\frac{dP}{dt} = kP \Rightarrow P = P_0 e^{kt}$$

(ii) Population with carrying capacity

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{K}\right) \quad (\text{logistic equation})$$

(iii) Newton's law of cooling

$$\frac{dT}{dt} = -k(T - T_a)$$

$$\Rightarrow T = T_a + (T_0 - T_a) e^{-kt}$$

(iv) RC Circuit

$$\frac{dQ}{dt} + \frac{1}{RC} Q = \frac{E}{R}$$

8. FURTHER TYPES (NONLINEAR)

(i) Equations solvable for x

$$\text{If } \frac{dx}{dy} = F(x, y)$$

(ii) Equations reducible to linear

By suitable substitution (e.g., $y = vx$)

(iii) Clairaut's equation

$$y = xp + f(p), \quad p = \frac{dy}{dx}$$

$$\text{General solution: } y = Cx + f(C)$$

$$\text{Singular solution: } x = -f'(p), \quad y = pf(p) - f(p)$$

9. SUMMARY TABLE

Type of Equation	Standard Form	Method
Linear	$\frac{dy}{dx} + P(x)y = Q(x)$	Integrating factor
Separable	$\frac{dy}{dx} = X(x)Y(y)$	Separate variables
Homogeneous	$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$	Put $y = vx$
Exact	$M dx + N dy = 0$	Check $M_y = N_x$, then $\phi = C$
Bernoulli	$\frac{dy}{dx} + P(x)y = Q(x)y^n$	Put $v = y^{1-n}$

10. GATE LEVEL PROBLEMS

(i) Solve $\frac{dy}{dx} + \frac{1}{x}y = x$

Sol:

$$I.F. = e^{\int \frac{1}{x} dx} = x$$

$$y x = \int x \cdot x dx + C = \frac{x^3}{3} + C$$

$$\Rightarrow y = \frac{x^2}{3} + \frac{C}{x}$$

Ans: $y = \frac{x^2}{3} + \frac{C}{x}$

(ii) Solve $(y+x)dx + (x-y)dy = 0$

Sol:

$$M = y+x, \quad N = x-y$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 1 \quad (\text{Exact})$$

$$\text{Let } \phi_x = y+x$$

$$\phi = xy + \frac{x^2}{2} + g(y)$$

$$\phi_y = x + g'(y) = x-y \Rightarrow g'(y) = -y$$

$$g(y) = -\frac{y^2}{2}$$

$$\Rightarrow xy + \frac{x^2}{2} - \frac{y^2}{2} = C$$

Ans: $xy + \frac{x^2}{2} - \frac{y^2}{2} = C$

(iii) Solve $\frac{dy}{dx} = y(1-x)$

Sol:

$$\frac{dy}{y} = (1-x) dx$$

$$\ln|y| = x - \frac{x^2}{2} + C$$

$$\Rightarrow y = C e^{x - x^2/2}$$

Ans: $y = C e^{x - x^2/2}$

(iv) Solve $\frac{dy}{dx} + y = xy^2$

Sol:

Divide by y^2 ,

$$y^{-2} \frac{dy}{dx} + y^{-1} = x$$

Put $v = y^{-1} \Rightarrow \frac{dv}{dx} = -y^{-2} \frac{dy}{dx}$

$$-\frac{dv}{dx} + v = x \Rightarrow \frac{dv}{dx} - v = -x$$

$$I.F. = e^{-x}$$

$$v e^{-x} = \int -x e^{-x} dx + C$$

$$= (x+1)e^{-x} + C$$

$$v = x+1 + C e^x \Rightarrow y = \frac{1}{x+1 + C e^x}$$

Ans: $y = \frac{1}{x+1 + C e^x}$

(v) Population model: $\frac{dP}{dt} = 0.1P$,

$$P(0) = 1000$$

Sol:

$$P = P_0 e^{kt} = 1000 e^{0.1t}$$

$$\text{At } t = 20,$$

$$P = 1000 e^{2} \approx 7389.06$$

Ans: $P \approx 7389$

11. SHORT TRICKS

- ✓ For linear equations, always find I.F. first.
- ✓ Exact equation: check $M_y = N_x$.
- ✓ In Bernoulli, divide by y^n and put $v = y^{1-n}$.
- ✓ For homogeneous, use $y = vx$.
- ✓ Do not forget $+C$ in the final answer.

12. COMMON MISTAKES

- ✗ Missing the integrating factor.
- ✗ Wrong differentiation while checking exactness.
- ✗ Forgetting to substitute back in Bernoulli equations.
- ✗ Ignoring domain where solution is valid.

13. QUICK REFERENCE FORMULAS

- I.F. = $e^{\int P(x) dx}$
- Exact: $M dx + N dy = 0$, $M_y = N_x$
- Bernoulli: $v = y^{1-n}$, $\frac{dv}{dx} + (1-n)Pv = (1-n)Q$
- Separable: $\frac{dy}{Y(y)} = X(x) dx$
- Homogeneous: put $y = vx$

14. KEY REMINDERS

- ★ Understand the type of equation first.
- ★ Choose the correct method.
- ★ Practice GATE previous year questions.
- ★ Accuracy in algebra gives speed.
- ★ Solve more problems daily!



HIGHER-ORDER DIFFERENTIAL EQUATIONS

Constant Coefficients, Complementary Function,
Particular Integral and Initial Conditions

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1. INTRODUCTION

A differential equation involving the n th derivative of $y(x)$ is called an n -th order differential equation.

General form (Linear):

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = R(x)$$

where a_i are constants, $a_n \neq 0$.

If $R(x) = 0$, the equation is homogeneous; otherwise non-homogeneous.

2. LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = R(x)$$

where a_i are constants, is called LDE with constant coefficients.

Associated (Auxiliary) Equation:

$$a_n m^n + a_{n-1} m^{n-1} + \dots + a_1 m + a_0 = 0$$

The roots of this equation help to find the Complementary Function (C.F.).

3. COMPLEMENTARY FUNCTION (C.F.)

Let the roots of the auxiliary equation be:

(i) Distinct real roots: $m_1, m_2, \dots, m_n$

$$C.F. = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

(ii) Repeated real roots: m (repeated r times)

$$C.F. = (C_1 + C_2 x + C_3 x^2 + \dots + C_r x^{r-1}) e^{mx}$$

(iii) Complex roots: $\alpha \pm j\beta$ (repeated r times)

$$C.F. = e^{\alpha x} \left[(C_1 \cos \beta x + C_2 \sin \beta x) + x (C_3 \cos \beta x + C_4 \sin \beta x) + \dots + x^{r-1} (C_{2r-1} \cos \beta x + C_{2r} \sin \beta x) \right]$$

$\therefore$ C.F. contains n arbitrary constants.

4. PARTICULAR INTEGRAL (P.I.)

Method depends on $R(x)$.

(i) If $R(x) = e^{\lambda x} P_n(x)$, where $P_n(x)$ is a polynomial of degree n .

$$\text{Try P.I.} = x^s e^{\lambda x} Q_n(x)$$

where $Q_n(x)$ is a polynomial of degree n and s = multiplicity of λ in A.E.

(ii) If $R(x) = \sin ax, \cos ax$

$$\text{Try P.I.} = A \cos ax + B \sin ax$$

(iii) If $R(x) = P_n(x)$ (polynomial)

$$\text{Try P.I.} = Q_n(x)$$

(iv) If $R(x) = e^{\lambda x} (A \cos ax + B \sin ax)$

$$\text{Try P.I.} = x^s e^{\lambda x} (A \cos ax + B \sin ax)$$

5. GENERAL SOLUTION

The general solution (G.S.) is

$$y(x) = C.F. + P.I.$$

It contains n arbitrary constants.

6. INITIAL CONDITIONS

If initial conditions are given at $x = x_0$,

$$y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \dots$$

substitute in G.S. to find the constants.

Then particular solution satisfying initial conditions is obtained.

7. EXAMPLE

$$\text{Solve } \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + 2y = e^x$$

Sol:

$$\text{A.E.: } m^2 - 3m + 2 = 0 = (m-1)(m-2)$$

$$\text{Roots: } m_1 = 1, m_2 = 2$$

$$C.F. = C_1 e^x + C_2 e^{2x}$$

$R(x) = e^x, \lambda = 1$ (a root of A.E. with multiplicity 1)

$$\text{Try P.I.} = x e^x$$

$$y_p' = e^x + x e^x, y_p'' = e^x + e^x + x e^x = 2e^x + x e^x$$

Substitute in DE:

$$(2e^x + x e^x) - 3(e^x + x e^x) + 2(x e^x) = e^x \quad (\text{True})$$

$$\rightarrow y = C_1 e^x + C_2 e^{2x} + x e^x$$

8. GATE LEVEL PROBLEMS

$$(i) \text{ Solve } \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} + 6y = e^{2x}$$

Sol:

$$\text{A.E.: } m^2 - 5m + 6 = 0 = (m-2)(m-3)$$

$$\text{Roots: } 2, 3$$

$$C.F. = C_1 e^{2x} + C_2 e^{3x}$$

$$R(x) = e^{2x}; \lambda = 2 \text{ (a root)}$$

$$\text{Try P.I.} = x e^{2x}$$

$$y_p' = e^{2x} + 2x e^{2x}$$

$$y_p'' = 2e^{2x} + 2e^{2x} + 4x e^{2x} = 4e^{2x} + 4x e^{2x}$$

Substitute in DE:

$$(4e^{2x} + 4x e^{2x}) - 5(e^{2x} + 2x e^{2x}) + 6(x e^{2x}) = e^{2x}$$

$$(4 - 5 + 6)x e^{2x} + (4 - 5)e^{2x} = e^{2x} \rightarrow e^{2x} = e^{2x} \quad \checkmark$$

$$y = C_1 e^{2x} + C_2 e^{3x} + x e^{2x}$$

$$(ii) \text{ Solve } \frac{d^2 y}{dx^2} + 4y = \cos 2x$$

Sol:

$$\text{A.E.: } m^2 + 4 = 0 \rightarrow m = \pm 2j$$

$$C.F. = C_1 \cos 2x + C_2 \sin 2x$$

$$R(x) = \cos 2x \text{ (resonance)}$$

$$\text{Try P.I.} = x(A \cos 2x + B \sin 2x)$$

After substitution,

$$A = \frac{1}{4}, B = 0$$

$$\text{P.I.} = \frac{x}{4} \cos 2x$$

$$y = C_1 \cos 2x + C_2 \sin 2x$$

$$+ \frac{x}{4} \cos 2x$$

$$(iii) \text{ Solve } \frac{d^3 y}{dx^3} - 3 \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} = x$$

Sol:

$$\text{A.E.: } m^3 - 3m^2 + 2m = m(m-1)(m-2) = 0$$

$$\text{Roots: } 0, 1, 2$$

$$C.F. = C_1 + C_2 e^x + C_3 e^{2x}$$

$$R(x) = x \text{ (polynomial of degree 1)}$$

$$\text{Try P.I.} = Ax + B$$

$$\text{Substitute: } 0 - 3(0) + 2(A) = x$$

$$\rightarrow 2A = 1 \rightarrow A = \frac{1}{2}$$

$$\text{P.I.} = \frac{x}{2} + B$$

(B is absorbed in C_1)

$$y = C_1 + C_2 e^x + C_3 e^{2x} + \frac{x}{2}$$

$$(iv) \text{ Solve } \frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + 2y = e^{-x} \cos x$$

Sol:

$$\text{A.E.: } m^2 + 2m + 2 = (m+1)^2 + 1 = 0$$

$$\text{Roots: } -1 \pm j$$

$$C.F. = e^{-x} (C_1 \cos x + C_2 \sin x)$$

$$R(x) = e^{-x} \cos x; \lambda = -1, a = 1$$

$$(-1 + j) \text{ is a root, multiplicity } = 1$$

$$\text{Try P.I.} = x e^{-x} (A \cos x + B \sin x)$$

$$\text{After substitution, } A = \frac{1}{2}, B = 0$$

$$\text{P.I.} = \frac{x}{2} e^{-x} \cos x$$

$$y = e^{-x} (C_1 \cos x + C_2 \sin x)$$

$$+ \frac{x}{2} e^{-x} \cos x$$

9. SHORT TRICKS

- ✓ Write auxiliary equation correctly.
- ✓ For repeated root of multiplicity r , use x^{r-1} factor.
- ✓ For $e^{\lambda x}$ term, if λ is a root of A.E. with multiplicity s , multiply P.I. by x^s .
- ✓ Always check substitution.

10. COMMON MISTAKES

- ✗ Forgetting multiplicity of roots.
- ✗ Wrong form of P.I. (missing x^2).
- ✗ Mistake in differentiating e^{ax} or $\sin, \cos$ terms.
- ✗ Not applying initial conditions correctly.

11. QUICK REFERENCE FORMULAS

- LDE with constant coefficients:
 $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_0 y = R(x)$
- A.E.: $a_n m^n + a_{n-1} m^{n-1} + \dots + a_0 = 0$
- G.S.: $y = C.F. + P.I.$
- No. of arbitrary constants = order of DE
- Apply initial conditions to find constants.

12. KEY REMINDERS

- ★ Practice more problems daily!
- ★ Identify the type of $R(x)$ first.
- ★ Accuracy in algebra = full marks.
- ★ Use initial conditions at the end to get particular solution.



ADVANCED DIFFERENTIAL EQUATIONS

Variation of Parameters, Cauchy-Euler Equations, Boundary-Value Problems

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1. VARIATION OF PARAMETERS

Consider the linear non-homogeneous ODE

$$y'' + p(x)y' + q(x)y = R(x)$$

Let $y_1(x)$, $y_2(x)$ be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

General solution is

$$y = C_1 y_1 + C_2 y_2 + y_p$$

Using variation of parameters,

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 , u_2 satisfy

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = R(x) \end{cases}$$

Let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ (Wronskian)

$$u_1' = -\frac{y_2 R(x)}{W}, \quad u_2' = \frac{y_1 R(x)}{W}$$

$$u_1 = \int -\frac{y_2 R}{W} dx, \quad u_2 = \int \frac{y_1 R}{W} dx$$

4. EXAMPLE - VARIATION OF PARAMETERS

Solve $y'' - y = e^x$ using variation of parameters.

Homogeneous equation: $y'' - y = 0$

Auxiliary equation: $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$y_1 = e^x, \quad y_2 = e^{-x}$$

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2$$

$$R(x) = e^x$$

$$u_1' = -\frac{y_2 R}{W} = -\frac{e^{-x} e^x}{-2} = \frac{1}{2}$$

$$u_2' = \frac{y_1 R}{W} = \frac{e^x e^x}{-2} = -\frac{e^{2x}}{2}$$

$$u_1 = \frac{x}{2}, \quad u_2 = -\frac{e^{2x}}{4}$$

$$y_p = u_1 y_1 + u_2 y_2 = \frac{x}{2} e^x - \frac{e^{2x}}{4} e^{-x} = \frac{x e^x}{2} - \frac{e^x}{4}$$

$$\text{General solution: } y = C_1 e^x + C_2 e^{-x} + e^x \left(\frac{x}{2} - \frac{1}{4} \right)$$

2. CAUCHY-EULER (EQUATION EQUATIONS)

A Cauchy-Euler equation has the form

$$x^2 y'' + a x y' + b y = R(x)$$

(a, b are constants)

(i) Homogeneous equation: $x^2 y'' + a x y' + b y = 0$

Try $y = x^m$

Substitute:

$$m(m-1) + a m + b = 0$$

$$\Rightarrow m^2 + (a-1)m + b = 0 \quad (\text{Auxiliary equation})$$

(ii) Non-homogeneous equation: $x^2 y'' + a x y' + b y = R(x)$

Method: Solve y_c from auxiliary equation.

Then find y_p depending on the form of $R(x)$.

If $R(x) = x^k$,

Try $y_p = A x^k$ (if k is not a root)

If $R(x) = x^k$ and k is a root m (multiplicity r),

Try $y_p = A x^k \ln x$ (if r = 1)

or $y_p = A x^k (\ln x)^r$ (if multiplicity r)

General solution: $y = y_c + y_p$

5. EXAMPLE - CAUCHY-EULER EQUATION

Solve $x^2 y'' - 3xy' + 2y = x^2$

Homogeneous equation: $x^2 y'' - 3xy' + 2y = 0$

Auxiliary: $m(m-1) - 3m + 2 = 0$

$$\Rightarrow m^2 - 4m + 2 = 0$$

$$\Rightarrow m = 2 \pm 1 \Rightarrow m = 1, 2$$

$$y_c = C_1 x + C_2 x^2$$

Since $R(x) = x^2$ and 2 is a root (m=2) of multiplicity 1,

Try $y_p = A x^2 \ln x$

Compute: $y_p' = A(2x \ln x + x)$

$$y_p'' = A(2 \ln x + 3)$$

Substitute in DE:

$$x^2 y_p'' - 3x y_p' + 2 y_p =$$

$$A [x^2 (2 \ln x + 3) - 3x(2x \ln x + x) + 2x^2 \ln x]$$

$$= A [2x^2 \ln x + 3x^2 - 6x^2 \ln x - 3x^2 + 2x^2 \ln x]$$

$$= -2A x^2 \ln x$$

Set equal to $x^2 \Rightarrow -2A x^2 \ln x = x^2 \Rightarrow A = -\frac{1}{2}$

$$y_p = -\frac{1}{2} x^2 \ln x$$

$$\text{General solution: } y = C_1 x + C_2 x^2 - \frac{1}{2} x^2 \ln x$$

3. BOUNDARY-VALUE PROBLEMS (BVP)

A BVP for a second-order linear ODE:

$$y'' + p(x)y' + q(x)y = r(x), \quad a \leq x \leq b$$

subject to boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta$$

Solution Procedure:

- Solve the differential equation to get general solution $y = C_1 y_1 + C_2 y_2 + y_p$.
- Apply boundary conditions to determine C_1 and C_2 .
- Obtain the particular solution satisfying the boundaries.

Note:

- Boundary conditions are specified at two different points (usually at $x = a$ and $x = b$).
- BVPs model physical problems like
 - vibrating strings,
 - heat conduction in a rod,
 - deflection of beams, etc.

6. EXAMPLE - BOUNDARY VALUE PROBLEM

Solve $y'' + y = 0$, $0 \leq x \leq \pi$ subject to

$$y(0) = 0, \quad y(\pi) = 0$$

General solution:

$$y = C_1 \cos x + C_2 \sin x$$

Apply $y(0) = 0$,

$$C_1 \cos 0 + C_2 \sin 0 = 0 \Rightarrow C_1 = 0$$

So, $y = C_2 \sin x$

Apply $y(\pi) = 0$,

$$C_2 \sin \pi = 0 \quad (\text{satisfied for any } C_2)$$

Particular (non-trivial) solution:

$$y(x) = C \sin x$$

(Trivial solution $y = 0$ also satisfies.)

7. GATE LEVEL PROBLEMS

(ii) Solve $y'' + y = \sec x$ using variation of parameters.

Sol: $y_c = C_1 \cos x + C_2 \sin x$

$$y_1 = \cos x, \quad y_2 = \sin x, \quad R(x) = \sec x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u_1' = -\frac{y_2 R}{W} = -\sin x \sec x = -\tan x$$

$$u_2' = \frac{y_1 R}{W} = \cos x \sec x = 1$$

$$u_1 = \int -\tan x dx = \ln |\cos x|$$

$$u_2 = \int 1 dx = x$$

$$y_p = u_1 y_1 + u_2 y_2 = (\ln |\cos x|) \cos x + x \sin x$$

$$\Rightarrow y = C_1 \cos x + C_2 \sin x + (\ln |\cos x|) \cos x + x \sin x$$

(iii) Solve $x^2 y'' + 3xy' + 2y = x$

Sol: Homogeneous: $m(m-1) + 3m + 2 = 0$

$$m^2 + 2m + 2 = 0 \Rightarrow m = -1 \pm i$$

$$y_c = \frac{1}{x} (C_1 \cos \ln x + C_2 \sin \ln x)$$

$R(x) = x$ (i.e., $k=1$ not a root)

Try $y_p = A x$

Substitute: $x^2(0) + 3x(A) + 2(Ax) = x$

$$5Ax = x \Rightarrow A = \frac{1}{5}$$

$$y_p = \frac{1}{5} x$$

$$y = \frac{1}{x} (C_1 \cos \ln x + C_2 \sin \ln x) + \frac{1}{5} x$$

(iii) Solve $y'' - 4y = e^{2x}$, $y(0) = 1$, $y'(0) = 0$

Sol: Homogeneous: $m^2 - 4 = 0$

$$m = \pm 2, \quad y_c = C_1 e^{2x} + C_2 e^{-2x}$$

$R(x) = e^{2x}$ (2 is a root, mult. 1)

Try $y_p = A x e^{2x}$

Solve: $y = (C_1 + A x) e^{2x} + C_2 e^{-2x}$

$$y(0) = 1 \Rightarrow C_1 + C_2 = 1$$

$$y' = (2C_1 + 2Ax + A) e^{2x} - 2C_2 e^{-2x}$$

$$y'(0) = 0 \Rightarrow 2C_1 + A - 2C_2 = 0$$

$$\text{Solve: } C_1 = \frac{1-A}{2}, \quad C_2 = \frac{1+A}{2}$$

$$0 = 2\left(\frac{1-A}{2}\right) + A - 2\left(\frac{1+A}{2}\right) = -2A$$

$$\Rightarrow A = 0, \quad C_1 = C_2 = \frac{1}{2}$$

$$\text{Therefore } y = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

(iv) Solve $x^2 y'' - xy' + y = 0$, $1 \leq x \leq e$

subject to $y(1) = 2$, $y(e) = 0$

Sol: $m(m-1) - m + 1 = 0 \Rightarrow m^2 - 2m + 1 = 0$

$$\Rightarrow m = 1 \quad (\text{repeated root})$$

$$y = (C_1 + C_2 \ln x) x$$

$$y(1) = 2 \Rightarrow C_1 = 2$$

$$y(e) = 0 \Rightarrow (2 + C_2) e = 0 \Rightarrow C_2 = -2$$

$$y = x(2 - 2 \ln x) = 2x(1 - \ln x)$$

8. SHORT TRICKS

- ✓ For variation of parameters, compute Wronskian carefully.
- ✓ For Cauchy-Euler, always try $y = x^m$.
- ✓ If $R(x) = x^k$ and k is root (mult. r), use $x^k (\ln x)^r$.
- ✓ Apply boundary conditions step-by-step to avoid errors.

9. COMMON MISTAKES

- ✗ Forgetting multiplicity while taking P.I. in Cauchy-Euler eqns.
- ✗ Wrong computation of Wronskian.
- ✗ Not applying boundary conditions properly in BVP.
- ✗ Sign mistakes in derivatives.

10. QUICK REFERENCE FORMULAS

- Variation of Parameters:

$$u_1' = -\frac{y_2 R}{W}, \quad u_2' = \frac{y_1 R}{W}$$
- Cauchy-Euler auxiliary: $m(m-1) + am + b = 0$
- BVP: Apply given conditions to G.S.
- General: $y = y_c + y_p$

11. KEY REMINDERS

- ★ Master variation of parameters for non-homogeneous DEs.
- ★ Cauchy-Euler frequently asked in GATE.
- ★ Practice BVP problems - very scoring!



PARTIAL DIFFERENTIAL EQUATIONS (PDEs)

Formation of PDEs, Standard Forms and Separation of Variables

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1. FORMATION OF PDEs

An equation involving unknown function of two or more independent variables and its partial derivatives is called a partial differential equation.

General form (two variables):

$$F(x, y, z, p, q, r, s, t, \dots) = 0$$

where,

$$z = z(x, y), \quad p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y},$$

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}, \dots$$

Examples:

(i) $z_x + z_y = x + y$

(ii) $x^2 z_{xx} + 2xy z_{xy} + y^2 z_{yy} = 0$

(iii) $z_{xx} + z_{yy} - 3z = e^x$

2. ORDER AND DEGREE OF A PDE

Order: The order of the highest order partial derivative present in the PDE.

Degree: The degree of the highest order partial derivative when the PDE is polynomial in the derivatives.

Examples:

(i) $(z_{xx})^2 + z_y = 0$

Order = 2, Degree = 2

(ii) $z_{xx} + (z_{xy})^3 + z = 0$

Order = 2, Degree = 3

(iii) $z_{xx} + \sin(z_{xy}) + z_y = 0$

Order = 2, Degree is not defined
(not polynomial in derivatives)

3. STANDARD FORMS OF SECOND ORDER LINEAR PDEs

A second-order linear PDE in two variables:

$$Az_{xx} + 2Bz_{xy} + Cz_{yy} + Dz_x + Ez_y + Fz = G$$

Classification based on $B^2 - AC$

(i) Elliptic : $B^2 - AC < 0$

Example: $z_{xx} + z_{yy} = 0$ (Laplace's equation)

(ii) Parabolic : $B^2 - AC = 0$

Example: $z_{xx} + 2z_{xy} = 0$ or $z_{xx} = 0$ (Heat equation)

(iii) Hyperbolic : $B^2 - AC > 0$

Example: $z_{xx} - z_{yy} = 0$ (Wave equation)

Type	Condition	Example
Elliptic	$B^2 - AC < 0$	$z_{xx} + z_{yy} = 0$
Parabolic	$B^2 - AC = 0$	$z_{xx} = 0$
Hyperbolic	$B^2 - AC > 0$	$z_{xx} - z_{yy} = 0$

4. SEPARATION OF VARIABLES - GENERAL IDEA

Assume solution of the form

$$z(x, y) = X(x)Y(y)$$

Substitute into PDE and separate variables so that one side is function of x only and other is function of y only. Both sides must be equal to a constant ($\pm \lambda$).

Example:

Solve $z_{xx} + z_{yy} = 0$

Assume $z = X(x)Y(y)$

$\Rightarrow X''Y + XY'' = 0$

$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$

$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$ (separation constant)

$\Rightarrow X'' - \lambda X = 0, \quad Y'' + \lambda Y = 0$

5. SOLUTION OF $X'' - \lambda X = 0$

(i) If $\lambda = 0$: $X'' = 0 \Rightarrow X = A_1 + A_2 x$

(ii) If $\lambda = k^2 > 0$: $X'' - k^2 X = 0$

$\Rightarrow X = A_1 e^{kx} + A_2 e^{-kx}$

(iii) If $\lambda = -k^2 < 0$: $X'' + k^2 X = 0$

$\Rightarrow X = A_1 \cos kx + A_2 \sin kx$

6. SOLUTION OF $Y'' + \lambda Y = 0$

(i) If $\lambda = 0$: $Y = B_1 + B_2 y$

(ii) If $\lambda = k^2 > 0$: $Y'' + k^2 Y = 0$

$\Rightarrow Y = B_1 \cos ky + B_2 \sin ky$

(iii) If $\lambda = -k^2 < 0$: $Y'' - k^2 Y = 0$

$\Rightarrow Y = B_1 e^{ky} + B_2 e^{-ky}$

7. GENERAL SOLUTION BY SEPARATION

$$z(x, y) = X(x)Y(y)$$

Let $\lambda = 0$: $z = (A_1 + A_2 x)(B_1 + B_2 y)$

Let $\lambda = k^2 > 0$:

$z = (A_1 e^{kx} + A_2 e^{-kx})(B_1 \cos ky + B_2 \sin ky)$

Let $\lambda = -k^2 < 0$:

$z = (A_1 \cos kx + A_2 \sin kx)(B_1 e^{ky} + B_2 e^{-ky})$

General solution is obtained by superposition (integral/series over λ).

8. EXAMPLES

(i) Solve $z_{xx} + z_{yy} = 0$

Sol:

Let $z = X(x)Y(y)$

$\Rightarrow X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = \lambda$

Case 1: $\lambda = k^2 > 0$

$z = (A_1 e^{kx} + A_2 e^{-kx})(B_1 \cos ky + B_2 \sin ky)$

Case 2: $\lambda = -k^2 < 0$

$z = (A_1 \cos kx + A_2 \sin kx)(B_1 e^{ky} + B_2 e^{-ky})$

Case 3: $\lambda = 0$

$z = (A_1 + A_2 x)(B_1 + B_2 y)$

General solution = Superposition of all cases.

(ii) Solve $z_{xx} = 0$

Sol:

Let $z = X(x)Y(y)$

$\Rightarrow X''Y = 0 \Rightarrow X'' = 0$

$\Rightarrow X = A_1 + A_2 x$

$z = (A_1 + A_2 x)Y(y)$

Substitute in PDE:

$(A_1 + A_2 x)Y'' = 0 \Rightarrow Y'' = 0$

$\Rightarrow Y = B_1 + B_2 y$

$\therefore z = (A_1 + A_2 x)(B_1 + B_2 y)$

(iii) Solve $z_{xx} - z_{yy} = 0$

Sol:

Let $z = X(x)Y(y)$

$\Rightarrow X''Y - XY'' = 0 \Rightarrow \frac{X''}{X} = \frac{Y''}{Y} = \lambda$

Case 1: $\lambda = k^2 > 0$

$z = (A_1 e^{kx} + A_2 e^{-kx})(B_1 e^{ky} + B_2 e^{-ky})$

Case 2: $\lambda = -k^2 < 0$

$z = (A_1 \cos kx + A_2 \sin kx)(B_1 \cos ky + B_2 \sin ky)$

Case 3: $\lambda = 0$

$z = (A_1 + A_2 x)(B_1 y + B_2)$

COMMON PHYSICAL PDEs

• Laplace's equation (steady state)

$z_{xx} + z_{yy} = 0$

• Heat equation (1-D steady)

$z_{xx} = 0$

• Wave equation in 2-D

$z_{xx} - z_{yy} = 0$

9. GATE LEVEL PROBLEMS

(i) Solve $z_{xx} + z_{yy} = 0$ such that $z(0, y) = \sin y, \quad z_x(0, y) = 0$.

Sol:

Using separation with $\lambda = k^2 > 0$

$z = (A \cosh kx + B \sinh kx)(C \cos ky + D \sin ky)$

From $z(0, y) = \sin y \Rightarrow AC \cos ky + AD \sin ky = \sin y$

$\Rightarrow AC = 0, \quad AD = 1 \Rightarrow C = 0, \quad D = \frac{1}{A}$

From $z_x(0, y) = 0 \Rightarrow BC \cos ky + BD \sin ky = 0$

$\Rightarrow B = 0$

$\therefore z = \left(\frac{1}{k} \sinh kx\right) \sin ky$

Superposition $\Rightarrow z = \sum_{n=1}^{\infty} \frac{1}{n} \sinh \frac{n\pi x}{L} \sin \frac{n\pi y}{L}$

(ii) Solve $z_{xx} = 0, \quad 0 \leq x \leq a, \quad z(0, y) = f(y), \quad z(a, y) = 0$.

Sol:

$z = (A_1 + A_2 x)Y(y)$

From $z(0, y) = f(y) \Rightarrow A_1 Y = f(y)$

From $z(a, y) = 0 \Rightarrow (A_1 + A_2 a)Y = 0$

$\Rightarrow A_1 + A_2 a = 0 \Rightarrow A_2 = -\frac{A_1}{a}$

$\Rightarrow z = A_1 \left(1 - \frac{x}{a}\right) Y(y)$

General solution depends on $Y(y)$.

(iii) Solve $z_{xx} - z_{yy} = 0, \quad 0 \leq x, y \leq L, \quad z(x, 0) = z(x, L) = 0, \quad z(0, y) = f(y), \quad z_x(L, y) = 0$.

Sol:

Using $\lambda = -k^2 < 0$:

$z = (A \cos kx + B \sin kx)(C e^{ky} + D e^{-ky})$

From $z(x, 0) = 0 \Rightarrow C + D = 0 \Rightarrow D = -C$

From $z(x, L) = 0 \Rightarrow C(e^{kL} - e^{-kL}) = 0 \Rightarrow C = 0$

(Trivial unless specific BCs lead to discrete k)

Apply remaining BCs to get $k = \frac{n\pi}{L}$, etc.

$\Rightarrow$ Use series in $\sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$

(iv) Solve $z_{xx} + z_{yy} = 0$ in $0 < x < a, 0 < y < b, \quad z(0, y) = z(a, y) = z(x, 0) = 0, \quad z(x, b) = f(x)$.

Sol:

$z = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{b}\right) \sin\left(\frac{n\pi x}{a}\right)$

Use $z(x, b) = f(x) \Rightarrow$

$f(x) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi b}{b}\right) \sin\left(\frac{n\pi x}{a}\right)$

$A_n = \left(\frac{2}{a} \sinh\left(\frac{n\pi b}{a}\right)\right)^{-1} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx$

10. SHORT TRICKS

- ✓ Assume $z = X(x)Y(y)$ for linear PDEs.
- ✓ Separate variables to get ODEs.
- ✓ Set each side = constant ($\pm \lambda$).
- ✓ Check all three cases $\lambda > 0, \lambda = 0, \lambda < 0$.
- ✓ Use superposition for general solution.

11. COMMON MISTAKES

- ✗ Forgetting negative sign in separation ($X''/X = -Y''/Y$ for sign PDE).
- ✗ Missing $\lambda = 0$ case.
- ✗ Not applying boundary conditions properly.
- ✗ Wrong classification ($B^2 - AC$ sign).

12. QUICK REFERENCE FORMULAS

- $\frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = -\lambda$ for $z_{xx} + z_{yy} = 0$
- $\frac{X''}{X} = \lambda, \quad \frac{Y''}{Y} = \lambda$ for $z_{xx} - z_{yy} = 0$
- $X'' = 0 \Rightarrow X = A_1 + A_2 x$
- $\lambda = k^2 > 0 \Rightarrow \cos, \sin$ or $e^{\pm kx}$
- $\lambda = -k^2 < 0 \Rightarrow e^{\pm ky}$

13. KEY REMINDERS

- ★ Understand physical meaning (Laplace, Heat, Wave).
- ★ Practice boundary-value problems.
- ★ Accuracy in algebra gives marks.
- ★ Draw region & apply BCs carefully.
- ★ Revise standard series forms.

COMPLEX VARIABLES - BASICS

Analytic Functions, Cauchy-Riemann Equations,
Harmonic Functions and Transformations

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1. COMPLEX NUMBERS - REVIEW

A complex number z is written as

$$z = x + iy$$

where $x, y \in \mathbb{R}$, $i = \sqrt{-1}$, $i^2 = -1$

$$\bar{z} = x - iy \quad (\text{complex conjugate})$$

$$|z| = \sqrt{x^2 + y^2} \quad (\text{modulus})$$

$$\text{Arg}(z) = \tan^{-1}\left(\frac{y}{x}\right) \quad (\text{principal value})$$

2. FUNCTIONS OF COMPLEX VARIABLE

Let $w = f(z) = u(x, y) + iv(x, y)$

where $u(x, y)$, $v(x, y)$ are real valued functions of x and y , with $z = x + iy$.

Limit: f is said to be defined (or has limit L) at z_0 if $\lim_{z \rightarrow z_0} f(z) = L$ exists.

Continuity: If u and v have continuous first partial derivatives in a neighbourhood of z_0 and f is analytic at $z_0 \Rightarrow f$ is continuous at z_0 .

3. COMPLEX DIFFERENTIABILITY - ANALYTIC FUNCTIONS

A function $f(z)$ is said to be analytic (or regular) at z_0 if $f'(z_0)$ exists.

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

f is analytic in a domain D if $f'(z)$ exists for every z in D .

Note: Every analytic function is infinitely differentiable in D .

4. CAUCHY-RIEMANN EQUATIONS (C-R)

If $f(z) = u(x, y) + iv(x, y)$ is analytic at $z = x + iy$, then u and v satisfy the C-R equations:

$$u_x = v_y$$

$$u_y = -v_x$$

where $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x = \frac{\partial v}{\partial x}$, $v_y = \frac{\partial v}{\partial y}$

Converse (sufficient condition):

If u_x, u_y, v_x, v_y are continuous in a region and satisfy C-R equations there, then $f(z)$ is analytic in that region.

5. ANALYTICITY TEST (PRACTICAL CHECK)

To test $f(z) = u + iv$ for analyticity in a region:

1. Compute u_x, u_y, v_x, v_y
2. Check C-R equations: $u_x = v_y, u_y = -v_x$
3. Verify continuity of u_x, u_y, v_x, v_y in the region

If all are satisfied $\Rightarrow f(z)$ is analytic.

6. HARMONIC FUNCTIONS

A real-valued function $\phi(x, y)$ is harmonic in a region if it satisfies Laplace's equation:

$$\nabla^2 \phi = \phi_{xx} + \phi_{yy} = 0$$

If $f(z) = u + iv$ is analytic, then

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

i.e., both u and v are harmonic functions.

7. PROPERTIES OF ANALYTIC FUNCTIONS

- ✓ Analytic $\Rightarrow$ infinitely differentiable.
- ✓ C-R equations hold at every point in the region.
- ✓ Zeros are isolated unless $f(z) \equiv 0$.
- ✓ If f and g are analytic $\Rightarrow f \pm g, fg, \frac{f}{g}$ ($g \neq 0$) are analytic.
- ✓ If f is analytic and $g'(z) = f(z) \Rightarrow g$ is analytic.

8. ELEMENTARY ANALYTIC FUNCTIONS

- Constant: $f(z) = C$ ($f'(z) = 0$)
- Identity: $f(z) = z$ ($f'(z) = 1$)
- Power: $f(z) = z^n$ ($f'(z) = n z^{n-1}$)
- Exponential: $f(z) = e^z$ ($f'(z) = e^z$)
- Trigonometric:
 $\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$
- Logarithm (principal):
 $\text{Log } z = \ln|z| + i \text{Arg } z \quad (-\pi < \text{Arg } z \leq \pi)$

9. CONFORMAL (ANGLE-PRESERVING) MAPPING

If $w = f(z)$ is analytic and $f'(z) \neq 0$ in a region, then f is conformal in that region.

Meaning: It preserves angles between two curves at the point.

(It may change lengths but not local shape or angles.)

10. EXAMPLES

(i) Show that $f(z) = z^2$ is analytic.

$$\text{Let } z = x + iy \Rightarrow f(z) = (x^2 - y^2) + i(2xy)$$

$$u = x^2 - y^2, \quad v = 2xy$$

$$u_x = 2x, \quad u_y = -2y, \quad v_x = 2y, \quad v_y = 2x$$

$$\text{Check: } u_x = v_y \quad (2x = 2x), \quad u_y = -v_x \quad (-2y = -2y)$$

Derivatives are continuous $\Rightarrow$ Analytic.

(ii) Show that $f(z) = e^z$ is analytic.

$$f(z) = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$u = e^x \cos y, \quad v = e^x \sin y$$

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y$$

$$v_x = e^x \sin y, \quad v_y = e^x \cos y$$

$$\text{Check: } u_x = v_y, \quad u_y = -v_x$$

$\Rightarrow$ Analytic everywhere.

(iii) Show that $u = x^2 - y^2$ is harmonic.

$$u_{xx} = \frac{\partial}{\partial x}(2x) = 2, \quad u_{yy} = \frac{\partial}{\partial y}(-2y) = -2$$

$$u_{xx} + u_{yy} = 2 + (-2) = 0$$

$\Rightarrow u$ is harmonic.

(iv) Show that $f(z) = \bar{z}$ is not analytic

$$f(z) = x - iy \Rightarrow u = x, \quad v = -y$$

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1$$

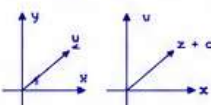
$$u_x = v_y? \quad (1 = -1) \quad \times$$

$\Rightarrow$ C-R equations not satisfied.
Hence, not analytic.

11. IMPORTANT CONFORMAL TRANSFORMATIONS

(i) $w = z + c$

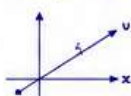
- Translation
- Shifts figure by vector c .



(ii) $w = kz$ ($k \neq 0$)

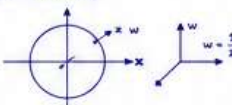
- Scaling by $|k|$
- Rotation by $\text{Arg } k$

$$\text{If } k = re^{i\theta}, \quad w = rze^{i\theta}$$



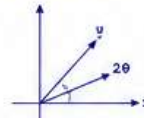
(iii) $w = \frac{1}{z}$

- Inversion in unit circle
- Inside $\leftrightarrow$ Outside
- Angle preserving



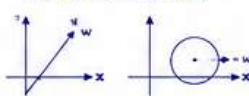
(iv) $w = z^2$

- Squares modulus
- Doubles argument



(v) $w = \frac{z-a}{z-b}$

- Maps circle/line to circle/line
- (Möbius transformation)



12. HARMONIC CONJUGATE

If u is harmonic, then there exists a function v (called harmonic conjugate) such that

$$u_x = v_y, \quad u_y = -v_x$$

Then $f(z) = u + iv$ is analytic.

Note: Harmonic conjugate is determined up to an additive constant.

13. COMMON MISTAKES

- ✗ Forgetting continuity of partial derivatives while applying C-R equations.
- ✗ Checking C-R equations only at a point, not in a region.
- ✗ Assuming differentiability without checking C-R equations.

14. QUICK REFERENCE FORMULAS

- C-R: $u_x = v_y, \quad u_y = -v_x$
- Harmonic: $u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$
- $(e^z)' = e^z, \quad (\sin z)' = \cos z, \quad (\cos z)' = -\sin z$
- $(\text{Log } z)' = \frac{1}{z} \quad (z \neq 0)$

15. KEY REMINDERS

- ★ Analytic $\Rightarrow$ C-R eqns + continuity.
- ★ Re(analytic) & Im(analytic) are harmonic.
- ★ Conformal mapping preserves angles.
- ★ Practice mapping problems for better understanding.



COMPLEX INTEGRATION

Cauchy's Integral Theorem, Cauchy's Integral Formula,
Taylor & Laurent Series

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1. COMPLEX INTEGRATION - INTRODUCTION

If $f(z) = u(x,y) + iv(x,y)$ is continuous on a simple closed contour C , then

$$\int_C f(z) dz = \int_C (u + iv)(dx + i dy) \\ = \int_C (u dx - v dy) + i \int_C (v dx + u dy)$$

C is positively oriented if traversed counter-clockwise.

Cauchy-Goursat theorem:

If $f(z)$ is analytic in a simply connected domain D , then $\int_C f(z) dz = 0$ for any closed contour C in D .

2. CAUCHY'S INTEGRAL THEOREM (C.I.T.)

If $f(z)$ is analytic in a simply connected region D , then for any closed contour C lying entirely in D ,

$$\int_C f(z) dz = 0$$

Consequences:

- ✓ The integral of an analytic function over any closed path in D is zero.
- ✓ The integral between two points is independent of the path.

3. CAUCHY'S INTEGRAL FORMULA (C.I.F.)

If $f(z)$ is analytic on and inside a simple closed contour C and z_0 lies inside C , then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

General form:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \\ (n = 0, 1, 2, \dots)$$

4. IMMEDIATE RESULTS FROM C.I.F.

$$\oint_C \frac{dz}{z - z_0} = 2\pi i$$

$$\oint_C \frac{dz}{(z - z_0)^{n+1}} = 0 \quad (n \geq 1)$$

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

$$\oint_C f(z) dz = 0 \quad (\text{if } z_0 \text{ is outside } C)$$

(C positively oriented simple closed contour)

5. EXAMPLES - USING C.I.T. & C.I.F.

(i) Evaluate $\oint_C \frac{dz}{z - 2}$, $C: |z - 2| = 3$
Pole $z_0 = 2$ is inside C

$$\therefore \oint_C \frac{dz}{z - 2} = 2\pi i$$

(ii) Evaluate $\oint_C \frac{dz}{z(z - 1)}$, $C: |z| = 2$

Poles: 0 (inside), 1 (inside)

$$\text{Res}_{z=0} = \lim_{z \rightarrow 0} \frac{1}{z - 1} = -1$$

$$\text{Res}_{z=1} = \lim_{z \rightarrow 1} \frac{1}{z} = 1$$

$$\therefore \oint_C \frac{dz}{z(z - 1)} = 2\pi i(-1 + 1) = 0$$

(iii) Evaluate $\oint_C \frac{e^z}{z - 1} dz$, $C: |z| = 2$

Pole at $z = 1$ (inside)

$$\therefore \oint_C \frac{e^z}{z - 1} dz = 2\pi i e^1 = 2\pi i e$$

6. TAYLOR SERIES

If $f(z)$ is analytic at z_0 , then it can be expressed as a Taylor series about z_0 in some neighbourhood:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

where

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

Radius of convergence R = distance from z_0 to the nearest singularity of $f(z)$.

7. EXAMPLES - TAYLOR SERIES

(i) Find the Taylor series of e^z about $z_0 = 0$.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \\ R = \infty \quad (\text{entire function})$$

(ii) Find Taylor series of $\frac{1}{1 - z}$ about $z_0 = 0$.

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots \\ |z| < 1 \rightarrow R = 1$$

(iii) Find Taylor series of $\sin z$ about $z_0 = 0$.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} z^{2n+1} \\ = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots, \quad R = \infty$$

8. LAURENT SERIES

If $f(z)$ is analytic in an annulus

$R_1 < |z - z_0| < R_2$, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \\ = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \quad (n \geq 0)$$

$$b_n = \frac{1}{2\pi i} \oint_C f(\zeta) (\zeta - z_0)^{n-1} d\zeta \quad (n \geq 1)$$

C : any circle with $R_1 < |z - z_0| < R_2$
(positively oriented)

9. EXAMPLES - LAURENT SERIES

(i) Find Laurent series of $\frac{1}{z}$ about $z_0 = 0$.

$$f(z) = \frac{1}{z} = \frac{1}{z - 0} = \frac{1}{z}$$

(Laurent series is only one term)

Region: $0 < |z| < \infty$

(ii) Find Laurent series of $\frac{1}{z - 1}$ about $z_0 = 0$.

$$\frac{1}{z - 1} = \frac{-1}{1 - z} = - \sum_{n=0}^{\infty} z^n \\ = -1 - z - z^2 - z^3 - \dots, \quad |z| < 1$$

(iii) Find Laurent series of $\frac{1}{z(z - 1)}$ about $z_0 = 0$.

$$\frac{1}{z(z - 1)} = -\frac{1}{z} + \frac{1}{1 - z} \\ = -\frac{1}{z} + \sum_{n=0}^{\infty} z^n, \quad 0 < |z| < 1$$

10. RELATIONSHIP BETWEEN SERIES

- Taylor series is a special case of Laurent series when the function is analytic at z_0 (i.e., no poles at z_0).
- Laurent series is useful when $f(z)$ has isolated singularities.

• Principal part of Laurent series:

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

• Regular part:

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

11. RESIDUE (COEFFICIENT OF $1/(z - z_0)$)

If $f(z)$ has a simple pole at z_0 , then

$$\text{Res } f(z) = b_1 = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

Generally, if

$$f(z) = \frac{\phi(z)}{\psi(z)}, \quad \phi(z_0) = 0, \quad \psi'(z_0) \neq 0$$

(simple pole)

$$\text{then } \text{Res } f(z) = \frac{\phi(z_0)}{\psi'(z_0)}$$

(Residues are widely used in evaluating real integrals)

12. EXAMPLE - RESIDUE & REAL INTEGRAL

$$\text{Evaluate } I = \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}$$

$$\text{Consider } f(z) = \frac{1}{z^2 + 1} = \frac{1}{(z - i)(z + i)}$$

Pole at $z = i$ in upper half plane.

$$\text{Res } f(z) = \lim_{z \rightarrow i} (z - i) \frac{1}{(z - i)(z + i)} = \frac{1}{2i}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} = 2\pi i \left(\frac{1}{2i} \right) = \pi$$

13. SHORT TRICKS

- ✓ Always check analyticity of $f(z)$ before using C.I.T. or C.I.F.
- ✓ For C.I.F., ensure z_0 is inside C .
- ✓ To find Taylor series, use derivatives at z_0 .
- ✓ To find Laurent series, expand in suitable region (annulus).
- ✓ Residue at simple pole $\rightarrow$ fast way to evaluate real integrals!

14. COMMON MISTAKES

- ✗ Applying C.I.T. where function is not analytic inside C .
- ✗ Forgetting orientation of contour.
- ✗ Choosing wrong radius/region for series.
- ✗ Ignoring poles while using Laurent series.
- ✗ Sign error in residues.

15. QUICK REFERENCE FORMULAS

- $\oint_C f(z) dz = 0$ (C.I.T.)
- $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$ (C.I.F.)
- $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$
- Taylor: $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$
- Laurent: $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

16. KEY REMINDERS

- ★ Understand analyticity first.
- ★ C.I.T. $\rightarrow$ integral over closed path = 0.
- ★ C.I.F. $\rightarrow$ value of function via contour.
- ★ Taylor series $\rightarrow$ inside radius R .
- ★ Laurent series $\rightarrow$ in annulus.
- ★ Practice residue problems regularly.

RESIDUE THEOREM & CONTOUR INTEGRALS

Residues, Poles, Evaluation of Contour Integrals
and GATE Problems

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1. POLES OF A FUNCTION

A point z_0 is a pole of $f(z)$ if $f(z)$ is not analytic at z_0 but $(z - z_0)f(z)$ is analytic at z_0 and $(z - z_0)f(z_0) \neq 0$.

Order of pole (m): smallest positive integer m such that $(z - z_0)^m f(z)$ is analytic at z_0 and $(z - z_0)^m f(z_0) \neq 0$.

Simple pole: Order = 1

Higher order pole: Order > 1

Essential singularity: not a pole.

2. RESIDUE OF A FUNCTION

If $f(z)$ has a pole of order m at z_0 , then the residue of f at z_0 is

$$\text{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$$

For simple pole ($m = 1$):

$$\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

3. RESIDUE THEOREM

If $f(z)$ is analytic in a domain D except for isolated singularities $z_1, z_2, \dots, z_n$ inside a positively oriented simple closed contour C lying in D , then

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f, z_k)$$

If no singularities inside $C \Rightarrow \text{Integral} = 0$

4. EVALUATION OF CONTOUR INTEGRALS

Step 1: Identify singular points of $f(z)$.

Step 2: Determine which singularities lie inside C .

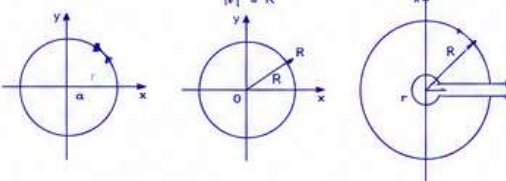
Step 3: Find residues at those points.

Step 4: Apply Residue Theorem.

Note: Orientation matters (clockwise $\Rightarrow -2\pi i$).

5. COMMON CONTOURS

(i) Circle: $|z - a| = r$ (ii) Circle centered at origin: $|z| = R$ (iii) Key Keyhole contour



6. KEY FORMULAS

• $\text{Res}\left(\frac{P(z)}{Q(z)}, z_0\right)$: poles are zeros of $Q(z)$.

• If $Q(z_0) = 0$, $Q'(z_0) \neq 0$ (simple pole), then

$$\text{Res}\left(\frac{P}{Q}, z_0\right) = \frac{P(z_0)}{Q'(z_0)}$$

• If pole of order m , use general formula (Sec. 2).

• Sum of residues (including at ∞) = 0

(Residue at $\infty = -\sum \text{residues at finite poles}$).

7. EXAMPLES - EVALUATE USING RESIDUE THEOREM

(i) $\oint_{|z|=2} \frac{dz}{z-1}$

Pole at $z = 1$ (simple) lies inside $|z| = 2$.

$$\text{Res} = \lim_{z \rightarrow 1} (z-1) \frac{1}{z-1} = 1$$

$$\therefore \text{Integral} = 2\pi i(1) = 2\pi i$$

(ii) $\oint_{|z|=2} \frac{z dz}{z^2 + 1}$

Poles: $z = i, -i$ (simple), both inside.

$$\text{Res at } z = i: \lim_{z \rightarrow i} \frac{z}{2z} = \frac{i}{2}$$

$$\text{Res at } z = -i: \lim_{z \rightarrow -i} \frac{z}{2z} = \frac{-i}{2}$$

$$\text{Sum} = 0 \Rightarrow \text{Integral} = 0$$

(iii) $\oint_{|z|=3} \frac{dz}{(z-1)(z-2)}$

Poles: $z = 1, 2$ (simple) inside.

$$\text{Res at } z = 1: \frac{1}{1-2} = -1$$

$$\text{Res at } z = 2: \frac{1}{2-1} = 1$$

$$\text{Sum} = 0 \Rightarrow \text{Integral} = 0$$

(iv) $\oint_C \frac{e^z}{z^2} dz$, where $C: |z| = 1$

Pole at $z = 0$ of order 2.

$$\text{Res} = \lim_{z \rightarrow 0} \frac{d}{dz} \left(z^2 \frac{e^z}{z^2} \right) = \lim_{z \rightarrow 0} e^z = 1$$

$$\text{Integral} = 2\pi i(1) = 2\pi i$$

8. GATE LEVEL PROBLEMS

(i) Evaluate $\oint_{|z|=1} \frac{dz}{z(z-1)}$

Poles: $z = 0, 1$ (simple). Inside $|z| = 1$ only $z = 0$.

$$\text{Res at } z = 0: \lim_{z \rightarrow 0} \frac{1}{z-1} = -1$$

$$\Rightarrow \text{Integral} = 2\pi i(-1) = -2\pi i$$

(ii) Evaluate $\oint_{|z|=3} \frac{z^2 + 1}{(z-1)(z-2)(z-3)} dz$

Poles: $1, 2, 3$ (simple). Inside: $1, 2$.

$$\text{Res at } z = 1: \frac{2}{(-1)(-2)} = 1$$

$$\text{Res at } z = 2: \frac{5}{(1)(-1)} = -5$$

$$\text{Sum} = -4 \Rightarrow \text{Integral} = 2\pi i(-4) = -8\pi i$$

(iii) Evaluate $\oint_C \frac{\sin z}{z - \pi} dz$,

where $C: |z| = 5$

Pole at $z = \pi$ (inside).

$$\text{Res} = \lim_{z \rightarrow \pi} (z - \pi) \frac{\sin z}{z - \pi} = \sin \pi = 0$$

$$\Rightarrow \text{Integral} = 0$$

(iv) Evaluate $\oint_{|z|=2} \frac{e^z}{z(z-2)(z-3)} dz$

Inside $|z| = 2$: only $z = 0$.

$$\text{Res at } z = 0: \lim_{z \rightarrow 0} \frac{e^z}{(z-2)(z-3)} = \frac{1}{(-2)(-3)} = \frac{1}{6}$$

$$\Rightarrow \text{Integral} = 2\pi i \left(\frac{1}{6} \right) = \frac{\pi i}{3}$$

9. RESIDUE AT INFINITY

If $f(z)$ has poles $z_1, z_2, \dots, z_n$ in the finite plane and $f(\infty) = 0$, then

$$\text{Res}(f, \infty) = - \sum_{k=1}^n \text{Res}(f, z_k)$$

Also,

$$\oint_{C_R} f(z) dz = -2\pi i \text{Res}(f, \infty), \text{ where}$$

$C_R: |z| = R$ (R large) positively oriented.

10. TAYLOR SERIES

If $f(z)$ analytic at z_0 ,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

Radius of convergence = distance from z_0 to nearest singularity.

11. LAURENT SERIES

If $f(z)$ analytic in $R_1 < |z - z_0| < R_2$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Principal part: $\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$
(terms with negative powers)

12. USING LAURENT SERIES (for integrals)

If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ in

$R_1 < |z - z_0| < R_2$, then

$$\oint_{|z-z_0|=r} f(z) dz = 2\pi i a_{-1}$$

(where $R_1 < r < R_2$)

13. SHORT TRICKS

✓ For simple pole at z_0 ,

$$\text{Res} = \lim_{z \rightarrow z_0} (z - z_0) f(z).$$

✓ For standard forms, factor quickly to locate poles.

✓ Check which poles lie inside contour.

✓ Sum residues only of enclosed poles.

✓ Be careful with orientation.

14. COMMON MISTAKES

✗ Forgetting poles at origin ($z = 0$).

✗ Including poles on the contour (not allowed).

✗ Wrong order of pole.

✗ Algebra mistakes in residue calculation.

✗ Orientation sign mistakes.

15. QUICK REFERENCE FORMULAS

• C.I.T.: $\oint_C f(z) dz = 0$ (no singularity inside)

• Res. Theorem: $\oint_C f(z) dz = 2\pi i \sum \text{Res}$

• Res (simple): $\text{Res}(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$

• Res (order m): $\frac{1}{(m-1)!} \lim_{z \rightarrow z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$

$$\oint_{|z-z_0|=r} \sum a_n (z - z_0)^n dz = 2\pi i a_{-1}$$

16. KEY REMINDERS

★ Residues convert tough integrals into simple algebra.

★ Find poles $\Rightarrow$ pick inside $\Rightarrow$ compute residues $\Rightarrow$ multiply by $2\pi i$.

★ Taylor series: for regular points.

★ Laurent series: for singular points.

★ Practice GATE problems daily!



PROBABILITY & STATISTICS

Probability, Conditional Probability, Random Variables,
Mean, Median, Mode, Variance & Standard Deviation

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1. BASIC PROBABILITY

- Sample space: S = set of all possible outcomes
- Event: $A \subseteq S$
- Probability of an event A :

$$P(A) = \frac{n(A)}{n(S)}$$

- $0 \leq P(A) \leq 1$
- $P(S) = 1$, $P(\emptyset) = 0$
- If A and B are mutually exclusive,
 $P(A \cup B) = P(A) + P(B)$
- If A and B are independent,
 $P(A \cap B) = P(A) P(B)$

2. CONDITIONAL PROBABILITY

If $P(B) > 0$, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- $P(A \cap B) = P(A|B) P(B) = P(B|A) P(A)$
- Law of total probability:
If $\{B_1, B_2, \dots, B_n\}$ is a partition of S , then

$$P(A) = \sum_{i=1}^n P(A|B_i) P(B_i)$$

3. BAYES' THEOREM

If $\{B_1, B_2, \dots, B_n\}$ is a partition of S with $P(B_i) > 0$, then

$$P(B_k|A) = \frac{P(A|B_k) P(B_k)}{\sum_{i=1}^n P(A|B_i) P(B_i)}$$

($k = 1, 2, \dots, n$)

4. RANDOM VARIABLES

A random variable X is a real-valued function defined on sample space S .

Types:

- (i) Discrete random variable - countable values
- (ii) Continuous random variable - any value in an interval

5. DISCRETE RANDOM VARIABLE

Probability mass function (pmf):
 $p(x) = P(X = x)$

Properties:

- (i) $p(x) \geq 0$ for all x
- (ii) $\sum_{all\ x} p(x) = 1$ (sum over all x)

6. CONTINUOUS RANDOM VARIABLE

Probability density function (pdf): $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x) dx = 1$$

$$P(a \leq X \leq b) = \int_a^b f(x) dx$$

7. MATHEMATICAL EXPECTATION (MEAN)

(i) Discrete: $\mu = E[X] = \sum x p(x)$

(ii) Continuous: $\mu = E[X] = \int_{-\infty}^{\infty} x f(x) dx$

Properties:

- $E[c] = c$
- $E[aX + b] = a E[X] + b$
- $E[X + Y] = E[X] + E[Y]$

8. VARIANCE & STANDARD DEVIATION

(i) Variance: $\sigma^2 = \text{Var}(X) = E[(X - \mu)^2]$

(ii) Standard deviation: $\sigma = \sqrt{\text{Var}(X)}$

For Discrete:

$$\sigma^2 = \sum (x - \mu)^2 p(x)$$

For Continuous:

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

Properties: $\text{Var}(aX + b) = a^2 \text{Var}(X)$

9. COMMON DISTRIBUTIONS

Distribution	pmf / pdf	Mean (μ)	Variance (σ^2)
Bernoulli(p)	$P(X=1)=p, P(X=0)=1-p$	p	$p(1-p)$
Binomial(n, p)	$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$	np	$np(1-p)$
Poisson(λ)	$P(X=k) = \frac{e^{-\lambda} \lambda^k}{k!}$	λ	λ
Normal(μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2

10. MEASURES OF CENTRAL TENDENCY

(i) Mean (Arithmetic Mean)

For data $x_1, x_2, \dots, x_n$

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

For frequency distribution

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

(ii) Median

Arrange data in ascending order.

• If n is odd: Median = $x_{(\frac{n+1}{2})}$

• If n is even:

$$\text{Median} = \frac{x_{(\frac{n}{2})} + x_{(\frac{n}{2} + 1)}}{2}$$

(iii) Mode

Value with highest frequency.

For grouped data,

$$\text{Mode} = l + \frac{(f_i - f_0)}{(2f_i - f_0 - f_2)} \times h$$

where l = lower limit of modal class,
 h = class width, f_i = freq. of modal class,
 f_0 = freq. of preceding class,
 f_2 = freq. of succeeding class.

11. RELATIONSHIPS

- Variance is always ≥ 0 .
- $\sigma \geq 0$
- If k is constant,
 $E[kX] = k E[X]$, $\text{Var}(kX) = k^2 \text{Var}(X)$
- For independent X, Y :
 $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$

12. EXAMPLES (QUICK REVISION)

(i) A fair die is thrown.

Find $P(\text{odd number})$.

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 3, 5\}$

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

(ii) Two coins are tossed.

Find $P(\text{at least one head})$.

$S = \{HH, HT, TH, TT\}$

$A = \{HH, HT, TH\}$

$$P(A) = \frac{3}{4}$$

(iii) Two events A and B with

$P(A) = 0.6, P(B) = 0.5$.

$P(A \cap B) = 0.2$.

Find $P(A|B)$.

$$P(A|B) = \frac{0.2}{0.5} = 0.4$$

(iv) If $X \sim \text{Bin}(n, p)$

Find $E[X]$ and $\text{Var}(X)$.

$$E[X] = np$$

$$\text{Var}(X) = np(1-p)$$

(v) Data: 2, 4, 4, 6, 8, 10

Mean = $\frac{34}{6} = 5.67$

Median = $\frac{4+6}{2} = 5$

Mode = 4

Variance = 6.22

Std. Dev. = 2.49

13. GATE LEVEL PRACTICE QUESTIONS

Q1. Two events A and B are such that
 $P(A) = 0.6, P(B) = 0.7, P(A \cap B) = 0.3$.
Find $P(A \cup B)$.

Q2. Three events E, F, G are mutually exclusive
with $P(E) = 0.2, P(F) = 0.5, P(G) = 0.3$.
Find $P(G | (E \cup F))$.

Q3. A bag contains 5 red and 3 blue balls.
Two balls are drawn without replacement.
Find $P(\text{both red})$.

14. GATE LEVEL PRACTICE QUESTIONS

Q4. If $X \sim \text{Poisson}(\lambda)$, show that
 $E[X] = \lambda$ and $\text{Var}(X) = \lambda$.

Q5. A random variable X has pdf

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(\frac{1}{4} < X < \frac{3}{4})$, $E[X]$, $\text{Var}(X)$.

Q6. For data: 3, 3, 4, 4, 4, 5, 7, 9
Find mean, median, mode and variance.

15. SHORT TRICKS

- ✓ For mutually exclusive events,
 $P(A \cup B) = P(A) + P(B)$
- ✓ For independent events,
 $P(A \cap B) = P(A) P(B)$
- ✓ Mean is sensitive to extreme values
- ✓ Median is positional
- ✓ Mode is most frequent value
- ✓ Larger variance $\rightarrow$ more spread

16. KEY REMINDERS

- ★ Practice formulas daily.
- ★ Understand concepts with examples.
- ★ Solve GATE PYQs on Probability & Statistics.
- ★ Accuracy in calculation is important.
- ★ Revise distributions & properties regularly.



DISTRIBUTIONS + FINAL REVISION

Binomial, Poisson, Normal Distributions, Sampling,
Correlation, Regression

+ Formula Sheet + Short Tricks + Common Mistakes
+ 10 Original GATE-Level MCQs with Solutions

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1. BINOMIAL DISTRIBUTION

Number of successes X in n independent Bernoulli trials (probability of success = p)
 $X \sim B(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$k = 0, 1, 2, \dots, n$$

$$\text{Mean } (\mu) = np$$

$$\text{Variance } (\sigma^2) = np(1-p)$$

$$\text{Standard Deviation } (\sigma) = \sqrt{np(1-p)}$$

2. POISSON DISTRIBUTION

Number of occurrences in a fixed interval of time/space with mean λ .

$$X \sim P(\lambda)$$

$$P(X = k) = \frac{e^{-\lambda} \lambda^k}{k!}$$

$$k = 0, 1, 2, \dots$$

$$\text{Mean } (\mu) = \lambda$$

$$\text{Variance } (\sigma^2) = \lambda$$

$$\sigma = \sqrt{\lambda}$$

3. NORMAL DISTRIBUTION

Continuous r.v. with bell-shaped curve.
 $X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty$$

$$\bullet \text{ Mean } (\mu) = \mu$$

$$\bullet \text{ Variance } (\sigma^2) = \sigma^2$$

$$\bullet \text{ Standard form: } Z = \frac{X - \mu}{\sigma} \sim N(0, 1)$$

$$\bullet P(a < X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

• Empirical rule:

$$P(|X - \mu| \leq \sigma) = 0.6826$$

$$P(|X - \mu| \leq 2\sigma) = 0.9544$$

$$P(|X - \mu| \leq 3\sigma) = 0.9973$$

4. SAMPLING & SAMPLING DISTRIBUTION

• Population mean = μ , variance = σ^2

• Sample size = n , Sample mean = $\bar{X}$

$$\bullet E[\bar{X}] = \mu$$

$$\bullet \text{Var}(\bar{X}) = \frac{\sigma^2}{n}$$

• If population is normal, then

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

• By CLT: for large n , $\bar{X}$ approx $N\left(\mu, \frac{\sigma^2}{n}\right)$

5. CORRELATION

Measures strength & direction of linear relationship between X and Y .

Population correlation coefficient (ρ):

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$-1 \leq \rho \leq 1$$

$\rho = 1$: Perfect positive

$\rho = -1$: Perfect negative

$\rho = 0$: No linear correlation

6. SIMPLE LINEAR REGRESSION

Model: $Y = a + bX + \epsilon$

$$b = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} = r \left(\frac{\sigma_Y}{\sigma_X} \right)$$

$$a = \bar{Y} - b\bar{X}$$

Regression line of Y on X :

$$\hat{Y} = a + bX$$

Regression line of X on Y :

$$\hat{X} = a' + b'Y$$

$$b' = r \left(\frac{\sigma_X}{\sigma_Y} \right), a' = \bar{X} - b'\bar{Y}$$

7. IMPORTANT FORMULA SHEET

$$\bullet \text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - \mu_X \mu_Y$$

$$\bullet \text{Var}(X) = E[X^2] - (E[X])^2$$

$$\bullet E[aX + b] = aE[X] + b$$

$$\bullet \text{Var}(aX + b) = a^2 \text{Var}(X)$$

$$\bullet \text{For independent } X, Y: \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

$$\bullet \text{If } X \sim N(\mu, \sigma^2), \text{ then } aX + b \sim N(a\mu + b, a^2\sigma^2)$$

$$\bullet Z = \frac{X - \mu}{\sigma} \Rightarrow X = \mu + \sigma Z$$

$$\bullet P(Z \leq z) = \Phi(z), P(Z > z) = 1 - \Phi(z)$$

$$\bullet \Phi(-z) = 1 - \Phi(z), \Phi(0) = 0.5$$

8. SHORT TRICKS

✓ For Binomial: $\frac{P(X = k+1)}{P(X = k)} = \frac{n-k}{k+1} \frac{p}{1-p}$

✓ For Poisson: $\frac{P(X = k+1)}{P(X = k)} = \frac{\lambda}{k+1}$

✓ For Normal: 68-95-99.7 Rule

✓ If $X \sim N(\mu, \sigma^2)$, standardize to Z

✓ Use symmetry: $\Phi(-z) = 1 - \Phi(z)$

✓ In regression, lines intersect at $(\bar{X}, \bar{Y})$

✓ If $\rho = 0 \Rightarrow$ lines are perpendicular

✓ Write formulas first, then substitute

9. COMMON GATE MISTAKES

- ✗ Forgetting conditions of distributions.
- ✗ Using σ instead of σ^2 in formulas.
- ✗ Wrong selection of n and p in Binomial.
- ✗ Confusing mean & variance.
- ✗ Not standardising in Normal problems.
- ✗ Sign mistakes in correlation/regression.
- ✗ Ignoring units in probability problems.
- ✗ Calculation errors in factorials and powers.
- ✗ Not reading the question carefully.
- ✗ Skipping rough work and checking.

10. 10 ORIGINAL GATE-LEVEL MCQs WITH DETAILED SOLUTIONS

Q1. If $X \sim B(10, 0.3)$, find $P(X \geq 4)$.

- (A) 0.3821 (B) 0.6179
(C) 0.6778 (D) 0.3222

Sol: $P(X \geq 4) = 1 - P(X \leq 3)$

$$= 1 - \sum_{k=0}^3 C(10, k) (0.3)^k (0.7)^{10-k}$$

$$= 1 - 0.3821 = 0.6179 \rightarrow (B)$$

Q2. If $X \sim P(3)$, find $P(X = 2)$.

- (A) 0.1494 (B) 0.2240
(C) 0.2986 (D) 0.3369

Sol: $P(X = 2) = \frac{1}{2!} e^{-3} 3^2$

$$= 1 - 0.8997 = 0.1003 \rightarrow (A)$$

Q3. If $X \sim N(50, 16)$, find $P(46 < X < 54)$.

- (A) 0.3829 (B) 0.6826
(C) 0.9544 (D) 0.9973

Sol: $Z = \frac{X - 50}{4}$

$$\text{Bounds: } -1 < Z < 1$$

$$P = 0.6826 \rightarrow (B)$$

Q5. For a population with $\mu = 100, \sigma = 20$, sample size $n = 25$. Find $\text{Var}(\bar{X})$.

- (A) 16 (B) 20 (C) 400 (D) 25

Sol: $\text{Var}(\bar{X}) = \frac{\sigma^2}{n} = \frac{20^2}{25} = 16 \rightarrow (A)$

Q6. If correlation coefficient $\rho = -0.6$, then the coefficient of determination is:

- (A) 0.36 (B) -0.6 (C) 0.6 (D) -0.36

Sol: $R^2 = \rho^2 = (-0.6)^2 = 0.36 \rightarrow (A)$

Q7. In regression of Y on X , $b = 2$ and $\bar{X} = 3, \bar{Y} = 7$. Find a .

- (A) 1 (B) 2 (C) 3 (D) 4

Sol: $a = \bar{Y} - b\bar{X} = 7 - 2(3) = 1 \rightarrow (A)$

Q9. If $X \sim B(20, 0.5)$, find $P(X \text{ is even})$.

- (A) 0.5000 (B) 0.5421
(C) 0.4579 (D) 1.0000

Sol: For $p = 0.5, P(X \text{ is even}) = P(X \text{ is odd}) = 0.5$

$$\rightarrow (A)$$

Q10. If $X \sim N(100, 400)$, find value of k such

that $P(X < k) = 0.8413$.

- (A) 120 (B) 130 (C) 140 (D) 150

Sol:

$$\Phi(1) = 0.8413 \Rightarrow \frac{(k - 100)}{20} = 1 \Rightarrow k = 120 \rightarrow (A)$$

STANDARD NORMAL TABLE ($\Phi(z)$)

z	0.00	0.01	0.02	0.03	0.04
0.0	0.5000	0.5040	0.5080	0.5120	0.5160
0.5	0.6915	0.6950	0.6985	0.7019	0.7054
1.0	0.8413	0.8438	0.8461	0.8885	0.8508
1.5	0.9332	0.9345	0.9357	0.9370	0.9382
2.0	0.9772	0.9778	0.9783	0.9788	0.9793
2.5	0.9938	0.9940	0.9942	0.9945	0.9947
3.0	0.9987	0.9987	0.9987	0.9988	0.9988

FINAL RAPID REVISION CHECKLIST

- ✓ Know all Distribution formulas.
- ✓ Memorize key properties.
- ✓ Practice standardization.

- ✓ Use tables smartly.
- ✓ Check units & assumptions.
- ✓ Avoid silly mistakes.

- ✓ Solve previous GATE papers.
- ✓ Time management is key.

- ✓ Stay calm & confident.
- ✓ Revision + Practice = Success!

"Well prepared today,

Confident tomorrow,

Topper forever!"

